Mechatronic Systems Design

MEC301

Asst. Prof. Dr.Ing.

Mohammed Nour A. Ahmed

mnahmed@eng.zu.edu.eg

https://mnourgwad.github.io

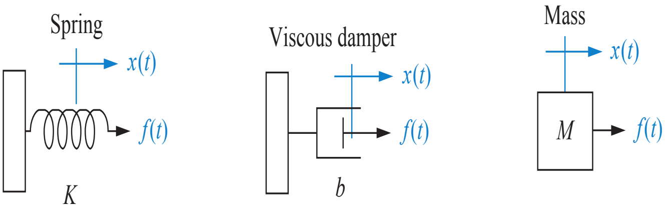
Lecture 5: **Modeling and Simulation 2**

Zagazig University | Faculty of Engineering | Computer and Systems Engineering Department | Zagazig, Egypt

Translational Mechanical System Transfer Functions

Component	Force–displacement	$G(s)$
Spring	$f(t) = K\,x$	K
Viscous damper	$f(t) = b\frac{dx(t)}{dt}$	$b\,s$
Mass	$f(t) = M\,\frac{d^2x(t)}{dt^2}$	$M\,s^2$

$$
\begin{aligned}
[f(t)] &= \text{N (newtons)},\\
[x(t)] &= \text{m (meters)},\\
[v(t)] &= \text{m/s (meters/second)},\\
[K] &= \text{N/m (newtons/meter)},\\
[b] &= \text{N s/m (newton.seconds/meter)},\\
[M] &= \text{kg (kilograms)}.
\end{aligned}
$$



Transitional Mechanical Systems

- Mechanical movements in a straight line (i.e. linear motion) are called “transitional”
- Basic Blocks are: Dampers, Masses, and Springs
- Springs represent the stiffness of the system
- Dampers (or dashpots) represent the forces opposing to the motion (i.e. friction)
- Masses represent the inertia

displacement

Force

$$\boldsymbol{F = kx}$$

velocity

$$\boldsymbol{F = cv}$$

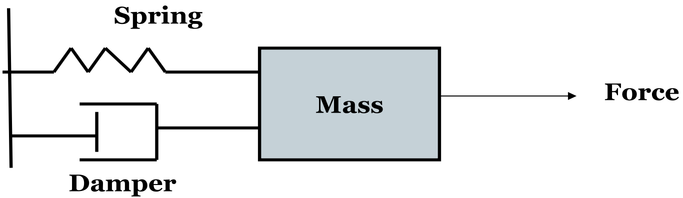
$$\boldsymbol{F = ma}$$

mass

acceleration

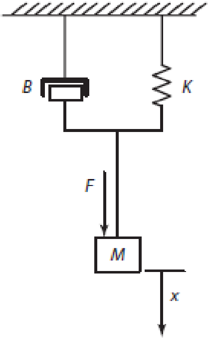
Transitional Mechanical Systems

- Equations for mechanical systems are based on Newton Laws
- Free body diagram

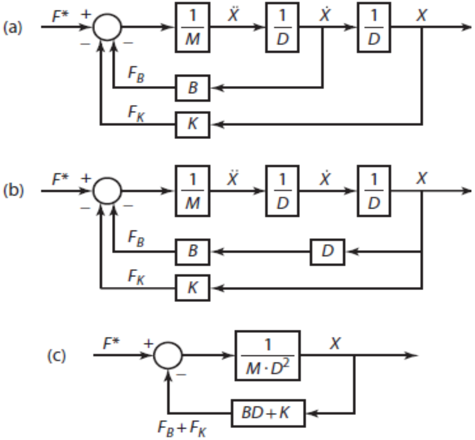


$$ma = F - kx - c\frac{dx}{dt}$$

Example: Mass-Spring-Damper

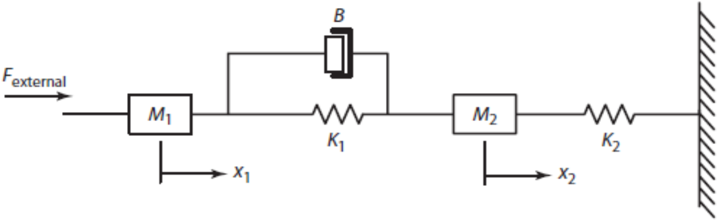


$$\ddot{x}(t) = -\frac{B}{M}\dot{x}(t) - \frac{K}{M}(x(t) - x_0) + \frac{1}{M}F(t)$$



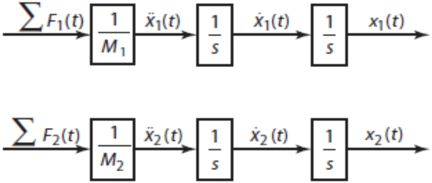
Note: D is Differentiation
1/D is Integration

Example: Two-Mass Mechanical System



$$\text{Mass 1:}\quad \ddot{x}_1(t) = \frac{1}{M_1}\sum F_1(t)$$

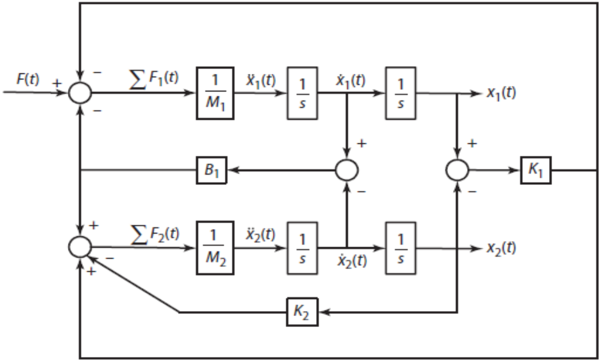
$$\text{Mass 2:}\quad \ddot{x}_2(t) = \frac{1}{M_2}\sum F_2(t)$$



Example: Two-Mass Mechanical System

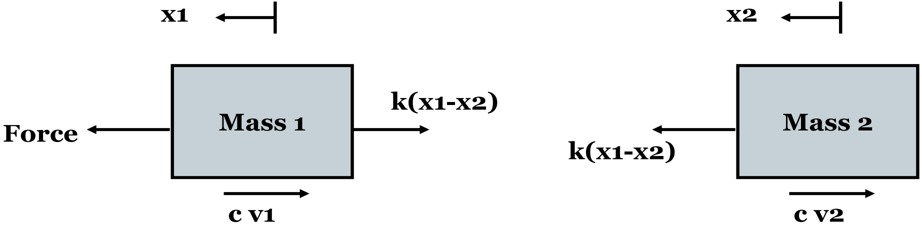
$$\sum F_1(t) = F_1(t) - K_1(x_1(t) - x_2(t)) - B(\dot{x}_1(t) - \dot{x}_2(t))$$

$$\sum F_2(t) = K_1(x_1(t) - x_2(t)) + B(\dot{x}_1(t) - \dot{x}_2(t)) - K_2x_2(t)$$



Example: Mechanical Model

- Consider a two carriage train system



$$m_1\ddot{x}_1 = f - k(x_1 - x_2) - c\dot{x}_1$$

$$m_2\ddot{x}_2 = k(x_1 - x_2) - c\dot{x}_2$$

Example continued

- Taking the Laplace transform of the equations gives

$$m_1 s^2 X_1(s) = F(s) - k(X_1(s) - X_2(s)) - csX_1(s)$$
$$m_2 s^2 X_2(s) = k(X_1(s) - X_2(s)) - csX_2(s)$$

  - Note: Laplace transforms the time domain problem into s-domain (i.e. frequency)

$$L\{x(t)\} = X(s) = \int_0^{\infty} e^{-st} x(t)dt$$
$$L\{\dot{x}(t)\} = sX(s)$$

Example continued

- Manipulating the previous two equations, gives the following transfer function (with F as input and V1 as output)

$$\frac{V_1(s)}{F(s)} = \frac{m_2 s^2 + cs + k}{m_1 m_2 s^3 + c(m_1 + m_2)s^2 + \left(km_1 + km_2 + c^2\right)s + 2kc}$$

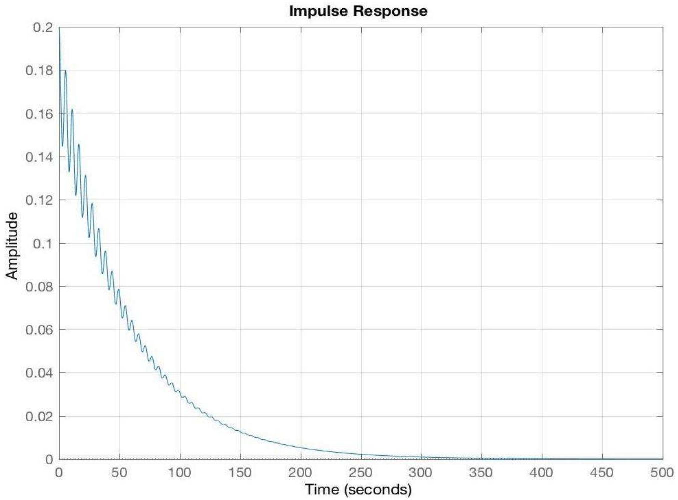
  - Note: Transfer function is a frequency domain equation that gives the relationship between a specific input to a specific output

Example continued

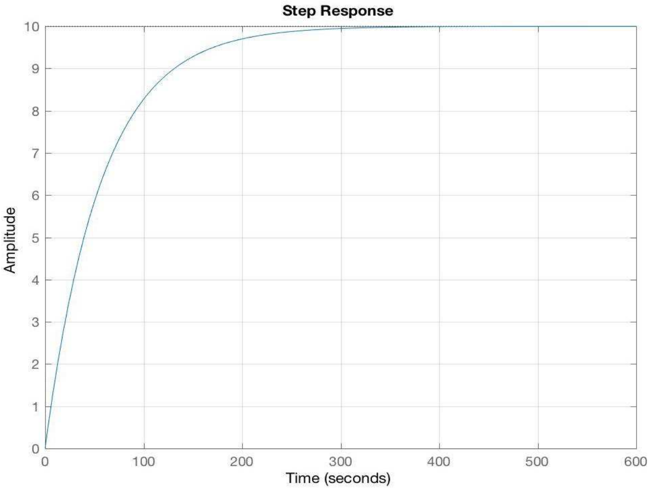
- Simulation using MATLAB

```
m1= 5; m2=0.7; k=0.8; c=0.05;

num=[m2 c k];
den=[m1*m2 c*m1+c*m2 k*m1+k*m2+c*c 2*k*c];
sys=tf(num,den); % constructs the transfer function

figure; impulse(sys); % plots the impulse response
grid on, box on;

figure; step(sys);  % plots the step response
grid on, box on;

figure; bode(sys); % plots the Bode plot
grid on, box on;
```

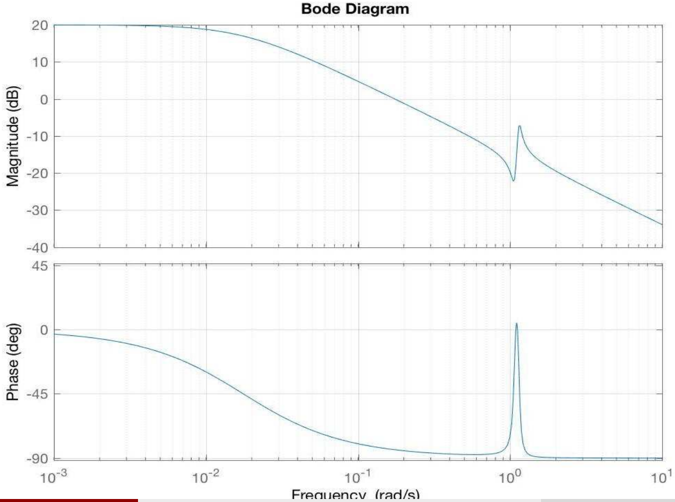
Example continued: Impulse response



Example continued: Step response

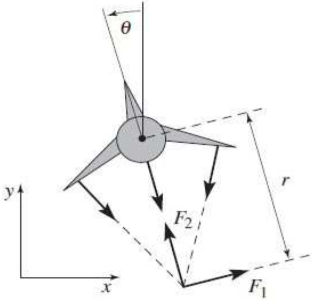


Example continued: Bode Plot



Example: Motion of Aircraft

(a) Harrier "jump jet"



(b) Simplified model

(x, y, θ) denote the position and orientation of the center of mass

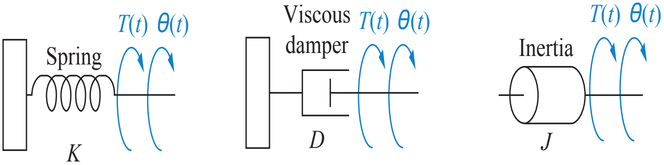
$$m\ddot{x} = F_1 \cos\theta - F_2 \sin\theta - c\dot{x},$$
$$m\ddot{y} = F_1 \sin\theta + F_2 \cos\theta - mg - c\dot{y},$$
$$J\ddot{\theta} = rF_1.$$

Rotational Mechanical System Transfer Functions

- Transfer functions of basic components

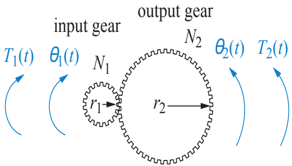
Component	Torque–angular displacement	$G(s)$
Spring	$T(t) = K\,\theta(t)$	K
Viscous damper	$T(t) = D\frac{d\theta(t)}{dt}$	$b\,s$
Inertia	$T(t) = J\,\frac{d^2\theta(t)}{dt^2}$	$J\,s^2$

$$
\begin{aligned}
[T(t)] &= \text{Nm (newtons.meters)},\\
[\theta(t)] &= \text{rad (radians)},\\
[\omega(t)] &= \text{rad/s (radians/second)},\\
[K] &= \text{Nm/rad (newtons.meter/rad)},\\
[D] &= \text{N m s/rad (newton.meter.seconds/ra}\\
[J] &= \text{kg m}^2\text{ (kilograms.meter}^2\text{)}.\\
Z &= \text{Rotational mechanical impedances}
\end{aligned}
$$



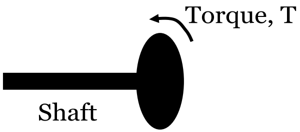
- Transfer functions for systems with gears

$$
\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}, \qquad \frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}, \qquad \frac{Z_2}{Z_1} = \left[\frac{N_2}{N_1}\right]^2
$$

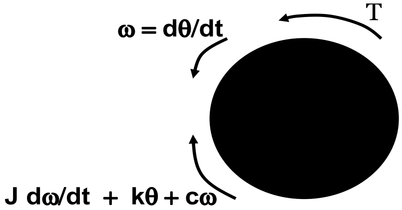


Rotational Mechanical Systems

- Consider a mechanical system that involves rotation



Side View



Top View

•The torque, T, replaces the force, F
•The angle, θ, replaces the displacement x
•The angular velocity, ω , replaces velocity v
•The angular acceleration, α, replaces the acceleration a
•The moment of inertia J, replaces the mass m

Rotational Mechanical Systems

- The mechanics equation becomes

damper coefficient

angular velocity

moment of inertia

angle

angular acceleration

spring coefficient

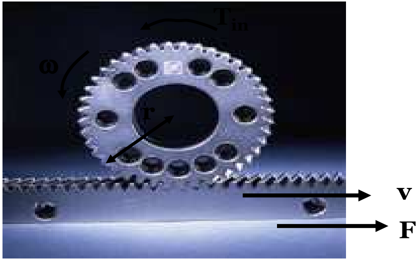
$$T = k\theta + c\omega + J\alpha$$

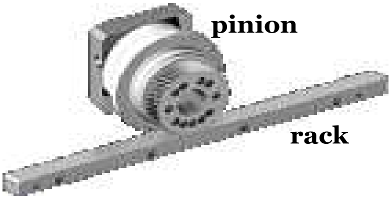
Torque

$$\Rightarrow T = k\theta + c\frac{d\theta}{dt} + J\frac{d^2\theta}{dt^2}$$

Example: Rotational-Transitional System

- Consider a rack-and-pinion system. The rotational motion of the pinion is transformed into transitional motion of the rack





For simplicity, the spring effects are ignored

$$T_{in} - T_{out} = J\frac{d\omega}{dt} + c_1\omega$$

Example continued

The rotational equation is

$$T_{in} - T_{out} = J\frac{d\omega}{dt} + c_1\omega$$

The transitional equation is

$$F - c_2 v = m\frac{dv}{dt}$$

Using the equations

$$T_{out} = rF$$
$$\omega = v/r$$

And manipulating the rotational and transitional equations with the input torque, Tin, as inputs and velocity, v, as output, we get

$$T_{in} = \left(c_1/r + c_2 r\right)v + \left(J/r + mr\right)\frac{dv}{dt}$$

Example continued

Let us take a look at the state space equations

In general,

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Cu}$$
$$\boldsymbol{y} = \boldsymbol{Bx} + \boldsymbol{Du}$$

where x is the states vector, y is the output vector, and u is the input vector

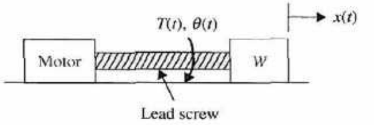
In our example, we will use the states: ω and v, the inputs: T_{in} and F the output: v

$$\begin{bmatrix} d\omega/dt \\ dv/dt \end{bmatrix} = \begin{bmatrix} -c_1/J & 0 \\ 0 & -c_2/m \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} + \begin{bmatrix} 1/J & -r/J \\ 0 & 1/m \end{bmatrix} \begin{bmatrix} T_{in} \\ F \end{bmatrix}$$

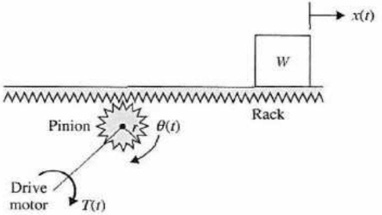
$$v = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix}$$

Manipulating the equations in the previous slide, we get

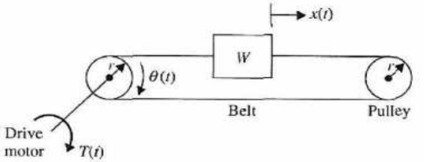
Conversion: Transitional and Rotational



$$J = \frac{W}{g}\left(\frac{L}{2\pi}\right)^2$$

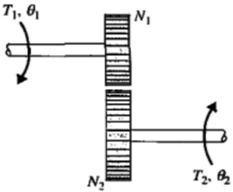


$$J = Mr^2 = \frac{W}{g}r^2$$

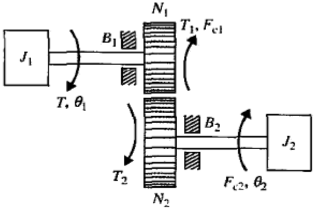


$$J = Mr^2 = \frac{W}{g}r^2$$

Gear Trains



$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}$$



Inertia: $\left(\frac{N_1}{N_2}\right)^2 J_2$

Viscous-friction coefficient: $\left(\frac{N_1}{N_2}\right)^2 B_2$

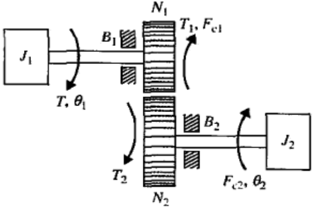
Torque: $\frac{N_1}{N_2} T_2$

Angular displacement: $\frac{N_1}{N_2} \theta_2$

Angular velocity: $\frac{N_1}{N_2} \omega_2$

Coulomb friction torque: $\frac{N_1}{N_2} F_{c2} \frac{\omega_2}{|\omega_2|}$

Gear Trains



$$T(t) = J_1 \frac{d^2\theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + F_{c1} \frac{\omega_1}{|\omega_1|} + T_1(t)$$

$$T_2(t) = J_2 \frac{d^2\theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + F_{c2} \frac{\omega_2}{|\omega_2|}$$

$$T_1(t) = \frac{N_1}{N_2} T_2(t) = \left(\frac{N_1}{N_2}\right)^2 J_2 \frac{d^2\theta_1(t)}{dt^2} + \left(\frac{N_1}{N_2}\right)^2 B_2 \frac{d\theta_1(t)}{dt} + \frac{N_1}{N_2} F_{c2} \frac{\omega_2}{|\omega_2|}$$

$$T(t) = J_{1e} \frac{d^2\theta_1(t)}{dt^2} + B_{1e} \frac{d\theta_1(t)}{dt} + T_F$$

where

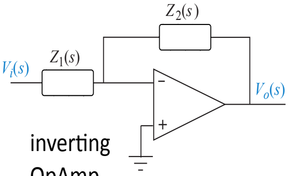
$$J_{1e} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 \qquad B_{1e} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2 \qquad T_F = F_{c1} \frac{\omega_1}{|\omega_1|} + \frac{N_1}{N_2} F_{c2} \frac{\omega_2}{|\omega_2|}$$

Electrical Network Transfer Functions

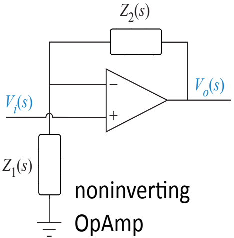
- Transfer functions of basic components

Symbol	Component	Voltage–current	$G(s)$
	Capacitor	$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$	$\frac{1}{Cs}$
	Resistor	$v(t) = R\, i(t)$	R
	Inductor	$v(t) = L \frac{di(t)}{dt}$	Ls

- Transfer functions of operational amplifiers
  - inverting operational amplifier: $G(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_2(s)}{Z_1(s)}$
  - noninverting operational amplifier: $G(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_1(s)+Z_2(s)}{Z_1(s)}$



inverting OpAmp



noninverting OpAmp

Electrical Systems: Basic Equations

- Resistor
  - Ohm's Law

Voltage ← $V = Ri$ → current, Resistance

- Inductor

$$V = L\frac{di}{dt}$$

Inductance

- Capacitor

$$V = \int \frac{1}{C} i\,dt$$

Capacitance

$$\Rightarrow i = C\frac{dV}{dt}$$

Power = Voltage x Current

Kirchoff Laws

- Equations for electrical systems are based on Kirchoff's Laws

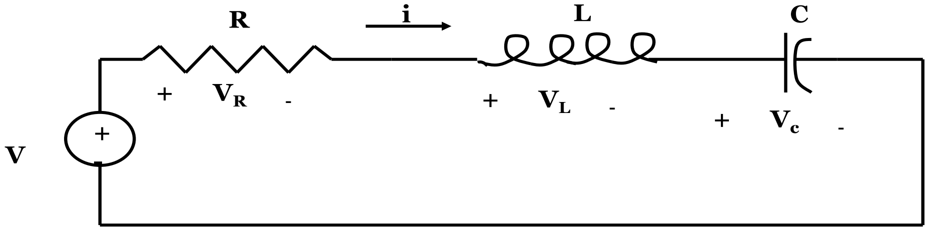
1. Kirchoff current law:

   Sum of Input currents at node = Sum of output currents

2. Kirchoff voltage law:

   Summation of voltage in closed loop equals zero

Example: RLC circuit



Using Kirchoff voltage law

$$V = Ri + L\frac{di}{dt} + \int \frac{1}{C} i dt \quad \text{Or} \quad V = Ri + L\frac{di}{dt} + V_c$$

since $i = C\dfrac{dV_c}{dt}$ Then

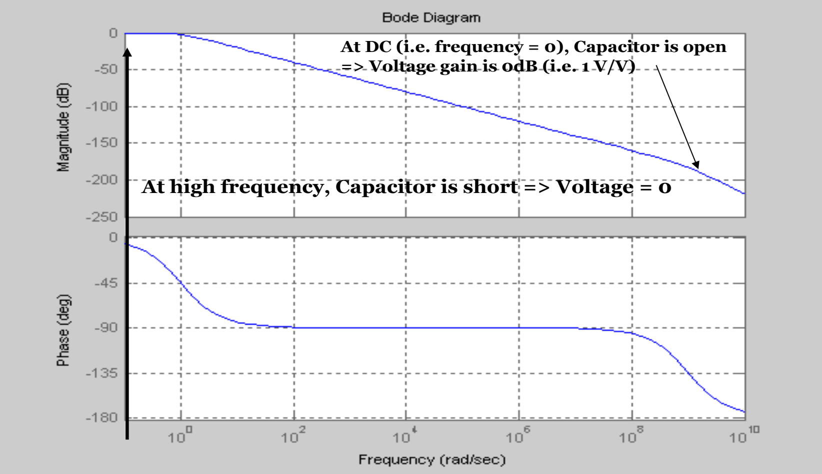
$$V = RC\frac{dV_c}{dt} + LC\frac{d^2V_c}{dt^2} + V_c$$

A second order differential equation

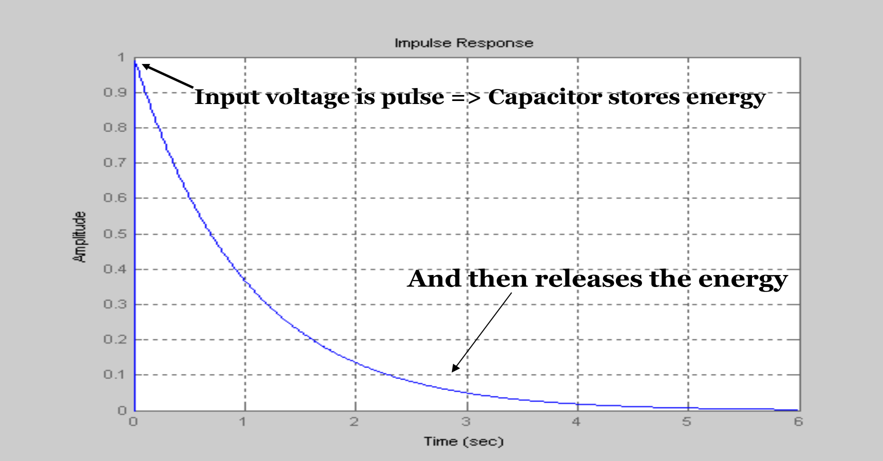
RLC MATLAB Code

- R=1000000; % R = 1MΩ
- L=0.001; % L=1 mH
- C=0.000001; % C= 1μF
- num=1; den=[L*C R*C 1];
- sys=tf(num,den);
- bode(sys)
- Impulse(sys)
- Step(sys)

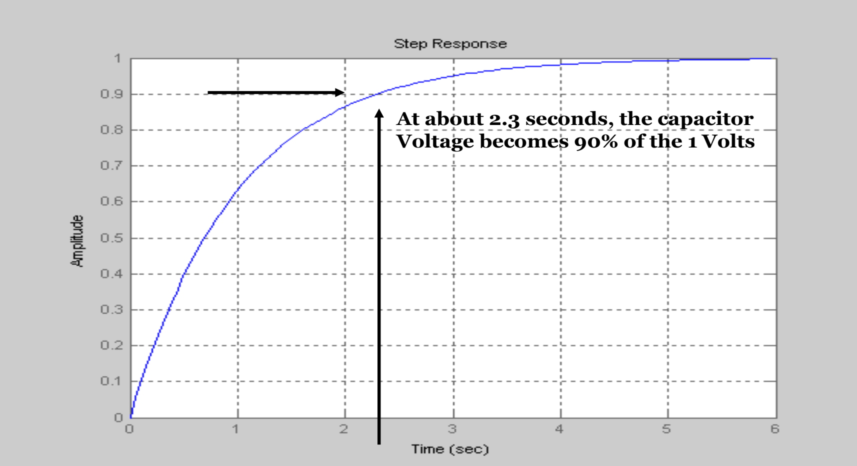
RLC Simulation: Bode Plot



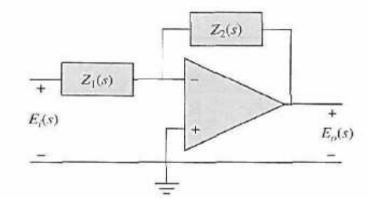
RLC Simulation: Impulse Response



RLC Simulation: Step Response



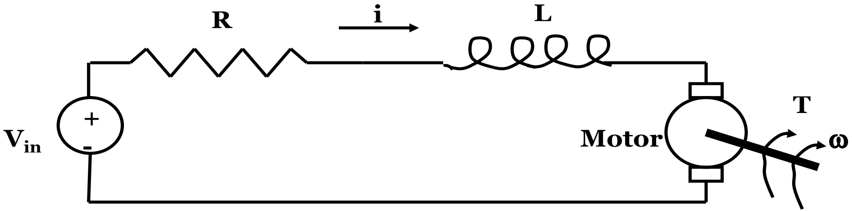
Op Amps



$$G(s) = \frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

Input Element	Feedback Element	Transfer Function	Comments
R_1, $Z_1 = R_1$	R_2, $Z_2 = R_2$	$-\frac{R_2}{R_1}$	Inverting gain, e.g., if $R_1 = R_2$, $e_o = -e_1$
R_1, $Z_1 = R_1$	C_2, $Y_2 = sC_2$	$\left(\frac{-1}{R_1 C_2}\right)\frac{1}{s}$	Pole at the origin, i.e., an integrator
C_1, $Y_1 = sC_1$	R_2, $Z_2 = R_2$	$(-R_2C_1)s$	Zero at the origin, i.e., a differentiator

PM-DC Motor Modeling



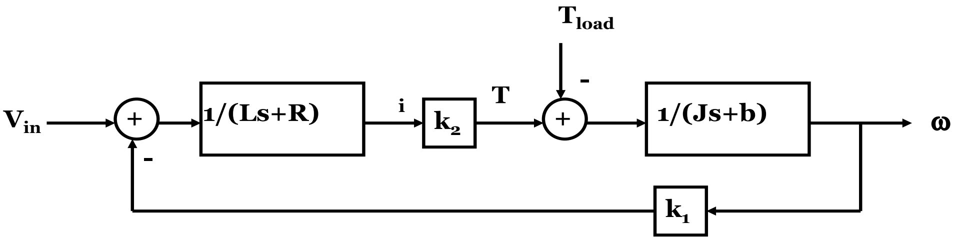
- The electrical equation is $$V_{in} = Ri + L\frac{di}{dt} + V_{emf}$$

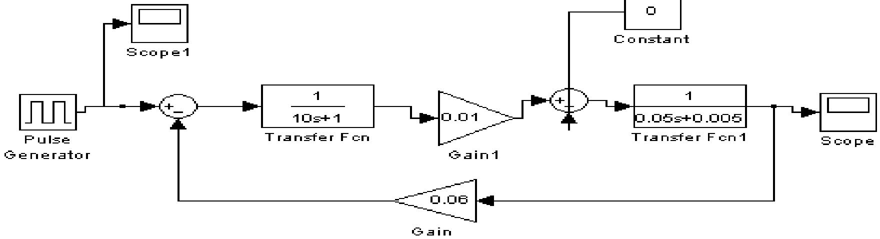
where V_{emf} (Back electromagnetic voltage) = $k_1\omega$

- The mechanical equation is $$T = J\frac{d\omega}{dt} + b\omega + T_{load}$$

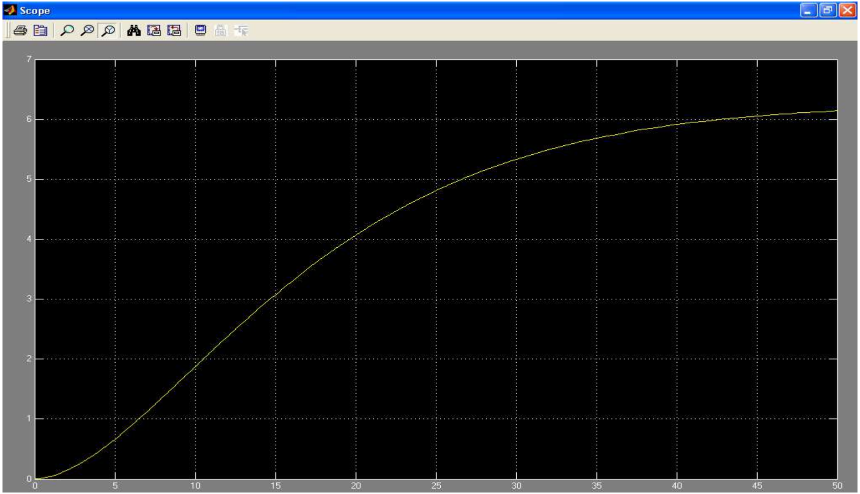
where $T = k_2 i$

DC Motor Model: Block Diagram





Simulation Result



Fluid Systems

- Fluid systems can be divided into two categories:
  - Hydraulic: fluid is a liquid and incompressible
  - Pneumatic: fluid is gas and can be compressed

- The volumetric rate of flow, q, is equivalent to the current
- The pressure difference, P_1-P_2, is equivalent to voltage

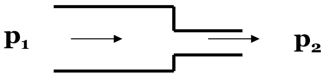
- The basic building blocks for hydraulic systems are: Hydraulic resistance, capacitance, and inertance

Hydraulic resistance

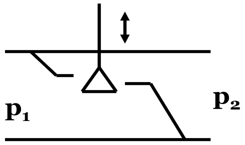
- Hydraulic resistance is the resistance to the fluid flow which occurs as a result of valves or pipe diameter changes

- The relationship between the volume rate of flow, q, and pressure difference, p_1-p_2 ,is given by Ohm's law

$$p_1 - p_2 = Rq$$



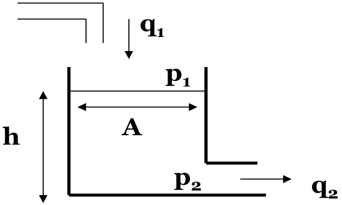
pipe



valve

Hydraulic Capacitance

- Potential energy stored in a liquid such as height of a liquid in a container



Volume Change

$$q_1 - q_2 = \frac{dV}{dt}$$

Volumetric rate of change

$$V = Ah \Rightarrow q_1 - q_2 = A\frac{dh}{dt}$$

Cross sectional Area

height

Hydraulic Capacitance

$$p_1 - p_2 = p = hg\rho$$

p → pressure, h → height, g → gravity, ρ → density

Note that $p = F/A = mg/A \Rightarrow p = \rho V g/A \Rightarrow p = hg\rho$

$$q_1 - q_2 = A\frac{dh}{dt} \Rightarrow q_1 - q_2 = A\frac{d\left(p/g\rho\right)}{dt} = \frac{A}{g\rho}\frac{dp}{dt}$$

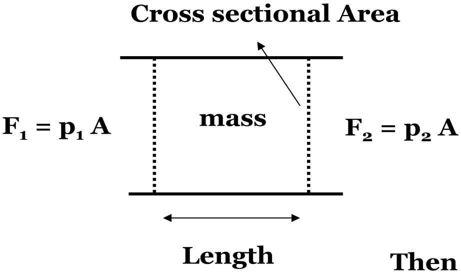
By letting the hydraulic capacitance be $C = \dfrac{A}{g\rho}$

We get $q_1 - q_2 = C\dfrac{dp}{dt}$

Hydraulic Inertance

- Equivalent to inductance in electrical systems
- To accelerate a fluid and increase its velocity a force is required

$$F_1 - F_2 = (p_1 - p_2)A$$



using

$$F_1 - F_2 = ma \Rightarrow m\frac{dv}{dt}$$

$$m = AL\rho$$

$$q = Av$$

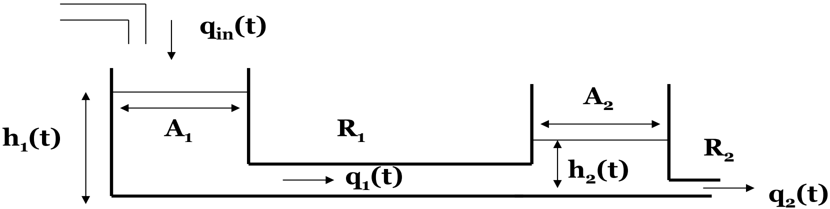
Then

$$p_1 - p_2 = I\frac{dq}{dt}$$

Where the Inertance is

$$I = \frac{L\rho}{A}$$

Hydraulic Example Modeling: an interactive 2-tank system



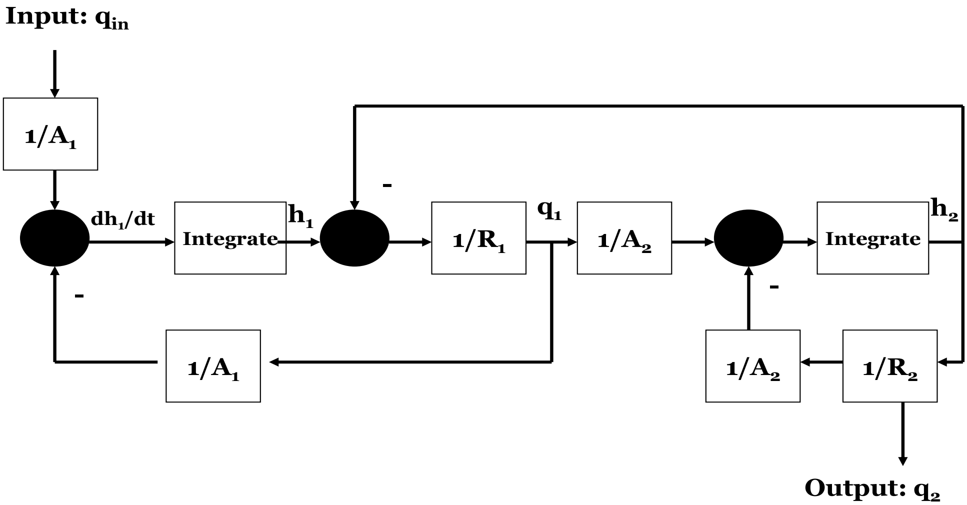
$$\frac{dh_1}{dt} = \left(q_{in}(t) - q_1(t)\right)/A_1$$

$$\frac{dh_2}{dt} = \left(q_1(t) - q_2(t)\right)/A_2$$

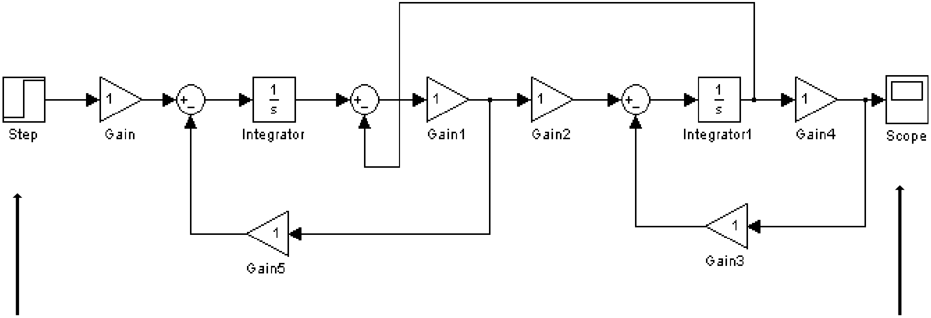
$$q_1(t) = \left(h_1(t) - h_2(t)\right)/R_1$$

$$q_2(t) = h_2(t)/R_2$$

Hydraulic Example Modeling: Block Diagram



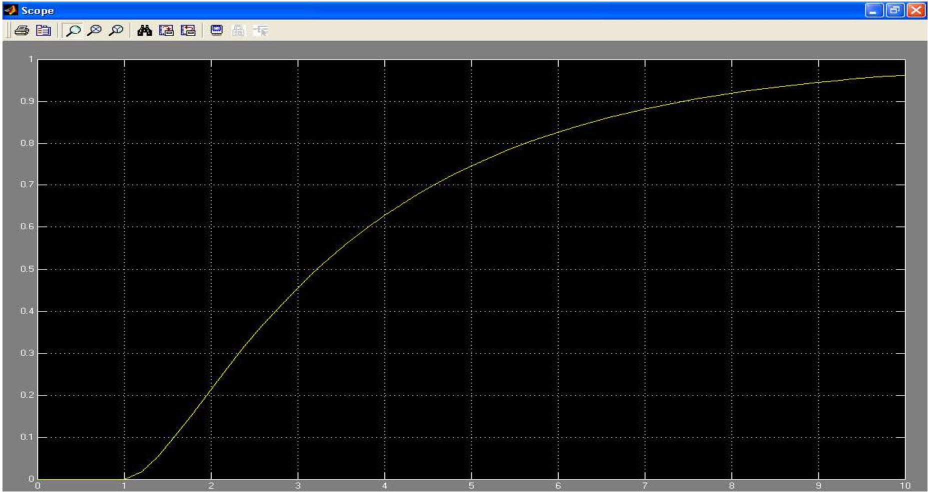
Hydraulic Example: Simulation



Input, q_{in}, is a step

Output, q_2, is taken to a virtual scope

Hydraulic Example: Simulation



Another Form of Analogies
Potential and Flow Variables

- When systems are in motion, the energy can be
  - *Increased by an energy-producing source outside the system*
  - *Redistributed* between components within the system
  - *Decreased by energy loss through components out of the* system.

- Therefore, a *coupled system becomes synonymous with energy transfer between systems.*

$$Potential\ Variable = PV$$
$$Flow\ Variable = FV$$

Analogies: FV and PV

	Flow Variable (FV)	Potential Variable (PV)
Electrical	Current	Voltage
Mechanical Transitional	Force	Velocity
Mechanical Rotational	Torque	Angular Velocity
Hydraulic	Volumetric Flow Rate	Pressure
Pneumatic	Mass Flow Rate	Pressure
Thermal	Heat Flow Rate	Temperature

Which Analogies to use?

- Force-Voltage makes more physical sense
  - Graphical Representation: Bond Graphs

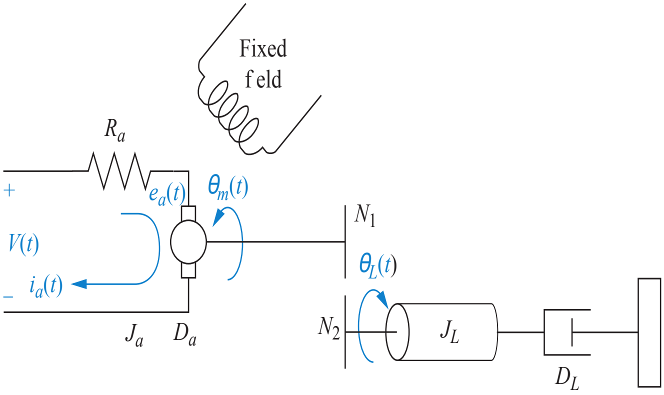
- Force-Current makes mathematical sense
- Sum of Currents= Zero and Sum of Forces = Zero
  - Graphical Representation: Linear Graphs

Conclusion

- Mathematical Modeling of physical systems is an essential step in the design process
- Simulation should follow the modeling in order to investigate the system response
- Mechatronic systems involve different disciplines and therefore an appropriate modeling technique to use is block diagrams
- Analogies among disciplines can be used to simplify the understanding of different dynamic behaviors

Electromechanical System Transfer Functions

DC Motor with Load



Electromechanical System Transfer Functions

DC Motor with Load

- for a DC motor, mechanical and electrical equations are:

$$V = R\,i + L\frac{di}{dt} + e_a \tag{1}$$

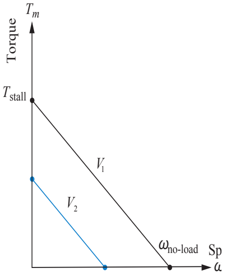
$$e_b = K_t\,\omega \tag{2}$$

$$T = K_t\,i = J_m\frac{d\omega}{dt} + D_m\,\omega + B \tag{3}$$

T	motor torque
K_t	torque constant
i	current,
V	supplied voltage,
ω	rotor speed,
e_b	back-emf ($e_b = K_e\omega$),
R, L	resistance and induction.

- For a fixed voltage, torque–speed curves are derived from (3) & (1):

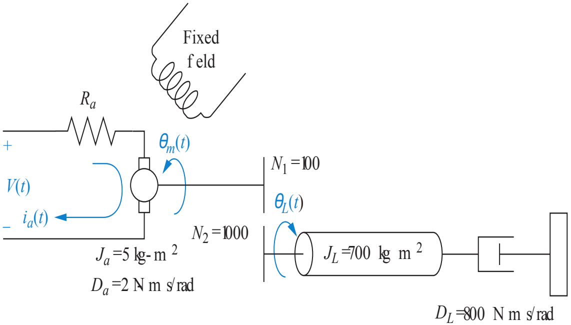
$$T = \frac{k_t}{R}(V - K_t\,\omega) = \frac{k_t}{R}V - k_m^2\,\omega \tag{4}$$

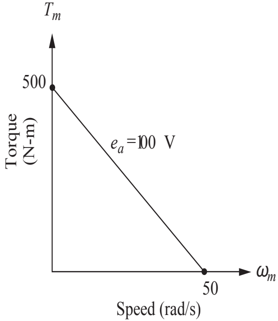


  - $K_m = \frac{k_t}{\sqrt{R}}$ is the **motor constant**, [numerically, $k_t == k_e$]
  - slope of the torque–speed curves is $-K_m^2$.
  - voltage-controlled DC motor has inherent damping in its mechanical behavior
  - torque increases in proportion to applied voltage and reduces as ω increases.

Example

Given the DC motor with load system and torque-speed curve, find the transfer function, $\theta_L(s)/V(s)$.





Electromechanical System Transfer Functions

DC Motor with Load

- to get the transfer function, we combine Laplace transforms of (1) through (3) and simplifying:

$$\frac{\theta_m(s)}{V(s)} = \frac{k_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]} \tag{5}$$

- the total inertia and damping at the armature of the motor are:

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12$$

$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10$$

- the electrical constants, K_t/R_a and K_b. From the torque-speed curve,

$$T_{stall} = 500, \qquad \omega_{no-load} = 50, \qquad V = 100$$

Electromechanical System Transfer Functions

DC Motor with Load

- Hence the electrical constants are:

$$\frac{K_t}{R_a} = \frac{T_{stall}}{V} = \frac{500}{100} = 5, \qquad K_b = \frac{V}{\omega_{no-load}} = \frac{100}{50} = 2$$

- Substituting system parameters into Eq.(5) yields:

$$\frac{\theta_m(s)}{V(s)} = \frac{5/12}{s\left[s + \frac{1}{12}\left(10 + 5 \times 2\right)\right]} = \frac{0.417}{s\left(s + 1.667\right)}$$

- to find the final transfer function (from the load–side, i.e. $\theta_L/V(s)$), we use the gear ratio, $N_1/N_2 = 1/10$, hence we get:

$$\frac{\theta_L(s)}{V(s)} = \frac{0.0417}{s\left(s + 1.667\right)}$$

Electromechanical System Transfer Functions

Car suspension system

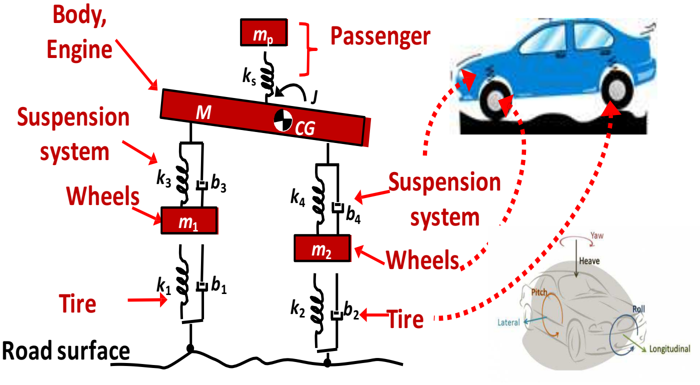
Example

Develop a model of an automobile which would be appropriate for studying the effectiveness of the **suspension system**, tire characteristics, and seat design on passenger comfort.

Electromechanical System Transfer Functions

Car suspension system

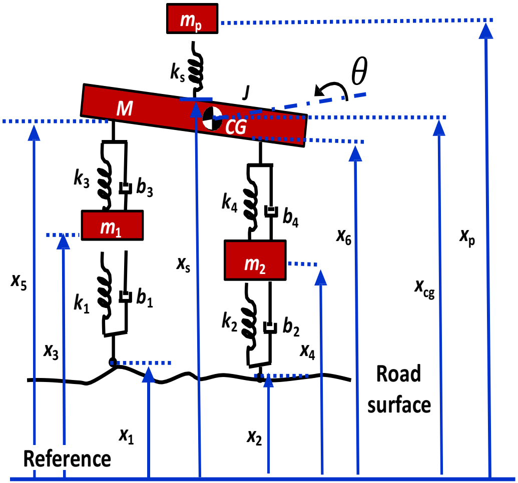
- For simplicity, neglect the side and roll motion
- An idealized model might be represented as:



Electromechanical System Transfer Functions

Car suspension system

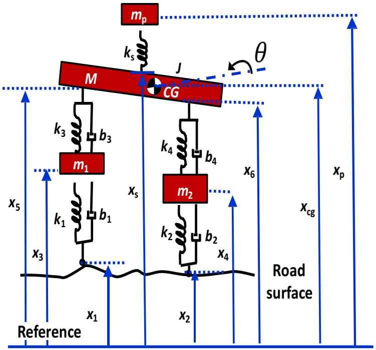
- An idealized model might be represented as:



Electromechanical System Transfer Functions

Car suspension system

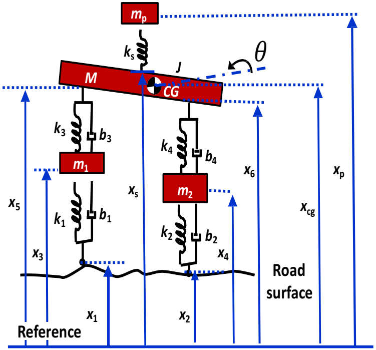
- System parameters are:
  - m_1 and m_2: wheels,
  - note: ($m_1 \neq m_2$) due to the suspensions are different
  - M and J: mass and pitching inertia of the main car body.
  - m_p: seat and passenger, k_s: for seat elasticity.
  - elasticity and energy dissipation properties of the tires are represented by k_1, k_2, b_1, and b_2.
  - note:$k_1 \neq k_2$ due to the pressure on the front $>$ Rear
  - suspension system is represented by k_3, k_4, b_3, and b_4.
- displacements x_1 and x_2 are inputs from the environment (road surface) and describing position of tires from Ref.
- x_3, x_4 are describing the position of center of the wheels from Ref.



Electromechanical System Transfer Functions

Car suspension system

- The goal is to develop a **mathematical model** to be able later **to control**.
- **No. of Equations = No. of masses** (m_1, m_2, m_p) and 2 more for M (linear and rotational) = 5 Ordinary Deferential Equations (ODE)
- For each mass (Linear motion): $\sum F_i = m_i a_i$
- For M only (Rotational motion): $\sum M_i = J\alpha$



Electromechanical System Transfer Functions

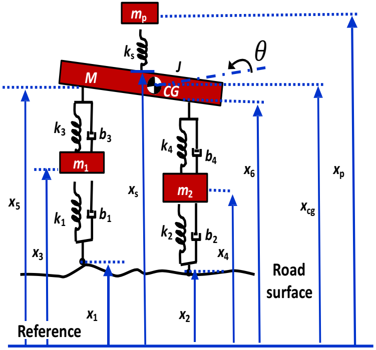
Car suspension system

- For front wheel mass m_1:

$$\begin{aligned} m_1\ddot{x}_3 &= -f_{k_1} - f_{b_1} - f_{k_2} - f_{b_2} \\ &= -k_1(x_3 - x_1) - b_1(\dot{x}_3 - \dot{x}_1) \\ &\quad - k_3(x_3 - x_5) - b_3(\dot{x}_3 - \dot{x}_5) \end{aligned} \tag{6}$$

- For rear wheel mass m_2:

$$\begin{aligned} m_2\ddot{x}_4 &= -f_{k_2} - f_{b_2} - f_{k_4} - f_{b_4} \\ &= -k_2(x_4 - x_2) - b_2(\dot{x}_4 - \dot{x}_2) \\ &\quad - k_4(x_4 - x_6) - b_4(\dot{x}_4 - \dot{x}_6) \end{aligned} \tag{7}$$



Electromechanical System Transfer Functions

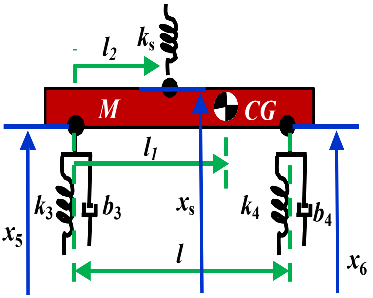
Car suspension system

- For body mass M:
  - Due to the linear motion

$$\begin{aligned} M\ddot{x}_{cg} &= -f_{k_3} - f_{b_3} - f_{k_4} - f_{b_4} - f_{k_s} \\ &= -k_3(x_5 - x_3) - b_3(\dot{x}_5 - \dot{x}_3) \\ &\quad - k_4(x_6 - x_4) - b_4(\dot{x}_6 - \dot{x}_4) \\ &\quad - k_s(x_s - x_p) \end{aligned} \tag{8}$$

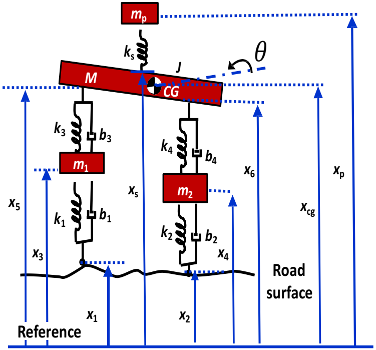
  - due to rotation: Assume the body under a small angle oscillation $(\cos\theta \approx 1, \sin\theta \approx \theta)$

$$\begin{aligned} J\ddot{\theta} &= -M_{k_3} - M_{b_3} - M_{k_4} - M_{b_4} - M_{k_s} \\ &= -l_1 k_3(x_5 - x_3) - l_1 b_3(\dot{x}_5 - \dot{x}_3) \\ &\quad - (l - l_1)k_4(x_6 - x_4) - (l - l_1)b_4(\dot{x}_6 - \dot{x}_4) \\ &\quad - (l_1 - l_2)k_s(x_s - x_p) \end{aligned} \tag{9}$$

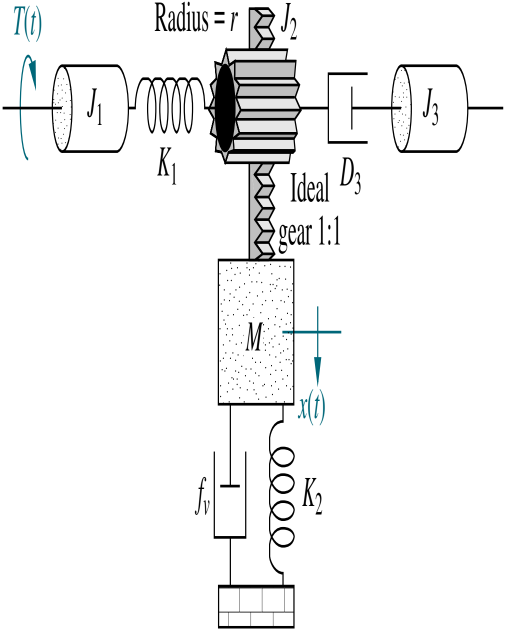


- in previous equation:
  - l_1: distance from the left end to center of gravity (CG),
  - l_2: distance to the seat mount,
  - l: total length (wheel base).
- For Passenger mass m_p:

$$
\begin{aligned}
m_p \ddot{x}_p &= -f_{k_s} \\
&= -k_s(x_p - x_s) \qquad (10)
\end{aligned}
$$



Given the combined translational and rotational system shown in Figure, find the transfer function, $G(s) = X(s)/T(s)$.



Writing the equations of motion,

$$(J_1s^2+K_1)\theta_1(s) \quad -K_1\theta_2(s) \quad = T(s)$$

$$-K_1\theta_1(s) + (J_2s^2+D_3s+K_1)\theta_2(s) + F(s)r \quad -D_3s\theta_3(s) = 0$$

$$-D_3s\theta_2(s) + (J_2s^2+D_3s)\theta_3(s) = 0$$

where $F(s)$ is the opposing force on J_2 due to the translational member and r is the radius of J_2.

for the translational member,

$$F(s) = (Ms^2+f_v s+K_2)X(s) = (Ms^2+f_v s+K_2)r\theta(s)$$

Substituting F(s) back into the second equation of motion,

$$(J_1 s^2+K_1)\theta_1(s) \qquad - K_1\theta_2(s) \qquad = T(s)$$

$$-K_1\theta_1(s) + [(J_2 + Mr^2)s^2+(D_3 + f_v r^2)s+(K_1 + K_2 r^2)]\theta_2(s) \qquad -D_3 s\theta_3(s) = 0$$

$$-D_3 s\theta_2(s) + (J_2 s^2+D_3 s)\theta_3(s) = 0$$

Notice that the translational components were reflected as equivalent rotational components by the

square of the radius.

square of the radius. Solving for $\theta_2(s)$, $\theta_2(s) = \dfrac{K_1(J_3s^2 + D_3s)T(s)}{\Delta}$, where Δ is the

determinant formed from the coefficients of the three equations of motion. Hence,

$$\frac{\theta_2(s)}{T(s)} = \frac{K_1(J_3s^2 + D_3s)}{\Delta}$$

Since

$$X(s) = r\theta_2(s), \quad \frac{X(s)}{T(s)} = \frac{rK_1(J_3s^2 + D_3s)}{\Delta}$$

Thanks for your attention.

Questions?

Asst. Prof. Dr.Ing.
Mohammed Nour A. Ahmed
mnahmed@eng.zu.edu.eg
https://mnourgwad.github.io

Robotics Research Interest Group (zuR^{2}IG)
Zagazig University | Faculty of Engineering | Computer and Systems Engineering Department | Zagazig, Egypt