Robotics

CSE4316

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Lecture 8: Robot Inverse Kinematics (cont.)

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Lecture: 8

Robot Inverse Kinematics (cont.)

- Geometric Approach
- **Algebraic Approach**

Algebraic Solution

Find the values of joint parameters that will put the tool frame at a desired position and orientation (within the workspace)

- Given the transformation matrix: $H = \begin{bmatrix} R_{3\times3} & T_{3\times1} \\ 0 & 1 \end{bmatrix}, R \in SO(3)$
- Find **all** solutions to: $T_n^0(q_1, \cdots, q_n) = H$
- Noting that: $T_n^0(q_1, \cdots, q_n) = A_1(q_1) \cdots A_n(q_n)$
- This gives **12** (nontrivial) equations with n unknowns
- with end effector position at $T_{3\times1}$
- and its orientation is obtained as:

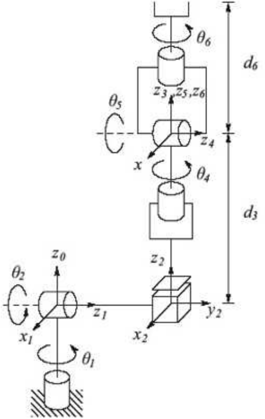
$$\psi = \text{atan2}\left(\frac{R_{21}}{R_{11}}\right), \quad \theta = \text{atan2}\left(-\frac{R_{31}\sin(\psi)}{R_{21}}\right), \quad \phi = \text{atan2}\left(\frac{R_{32}}{R_{33}}\right).$$

Robot Inverse Kinematics

Example: the Stanford manipulator

DH Parameters:

link	a_i	α_i	d_i	θ_i
1	0	-90	0	θ_1
2	0	90	d_2	θ_2
3	0	0	d_3	0
4	0	-90	0	θ_4
5	0	90	0	θ_5
6	0	0	d_6	θ_6



$d_2 = 0.154, d_6 = 0.263$

From which we get the transformation matrix (in general form):

$$H_6^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = [r_{ij}], \quad \begin{matrix} i = 1, \cdots, 4, \\ j = 1, \cdots, 4 \end{matrix}$$

Robot Inverse Kinematics

- For a given:

$$H_6^0 = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Find $\mathbf{q} = [\theta_1, \theta_2, d3, \theta_4, \theta_5, \theta_6]^T$.

- We have 12 **non-trivial** equations:

$$\begin{aligned}
c_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] - d_2\left(s_4c_5c_6 + c_4s_6\right) &= 0 \\
s_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] + c_1\left(s_4c_5c_6 + c_4s_6\right) &= 0 \\
-s_2\left(c_4c_5c_6 - s_4s_6\right) - c_2s_5c_6 &= 1 \\
c_1\left[-c_2\left(c_4c_5s_6 + s_4c_6\right) + s_2s_5s_6\right] - s_1\left(-s_4c_5s_6 + c_4c_6\right) &= 1 \\
-s_1\left[-c_2\left(c_4c_5s_6 - s_4c_6\right) - s_2s_5s_6\right] + c_1\left(-s_4c_5s_6 + c_4s_6\right) &= 0 \\
s_2\left(c_4c_5s_6 + s_4c_6\right) + c_2s_5s_6 &= 0 \\
c_1\left(c_2c_4s_5 + s_2c_5\right) - s_1s_4s_5 &= 0 \\
s_1\left(c_2c_4s_5 + s_2c_5\right) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6\left(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5\right) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6\left(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2\right) &= 0.763 \\
c_2d_3 + d_6\left(c_2c_5 - c_4s_2s_5\right) &= 0
\end{aligned}$$

- One solution: $\mathbf{q} = [\pi/2,\ \pi/2, 0.5,\ \pi/2, 0,\ \pi/2]^T$.
- next, we will see how to systematically find such solutions.

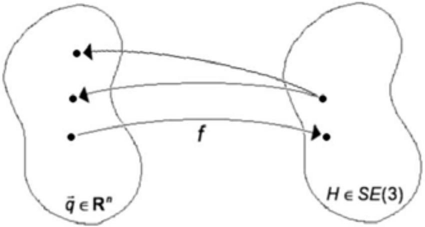
Robot Inverse Kinematics

- previous examples show how difficult it would be to obtain a closed-form solution to the 12 equations, instead,

> we develop systematic methods based upon the manipulator configuration

- For the forward kinematics there is always a unique solution
  - Potentially complex nonlinear functions

- The inverse kinematics may or may not have a solution
  - Solutions may or may not be unique
  - Solutions may violate joint limits
- Closed-form solutions are ideal !



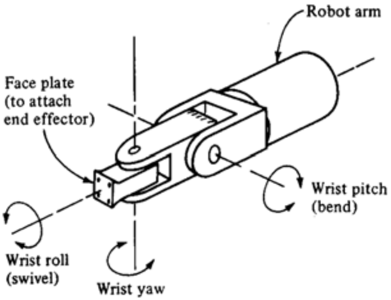
- Appropriate for systems that have an arm with a wrist
  - Such that the wrist joint axes are aligned at a point (the last 3 joint axes intersecting at a point)
- For such systems, we can decouple (split) the inverse kinematics problem into two parts:
  1. Inverse **position** kinematics: position of the wrist center
  2. Inverse **orientation** kinematics: orientation of the wrist
- First, assume 6DOF, the last three intersecting at O_c. For given R and O solve 9 rotational and 3 positional equations:

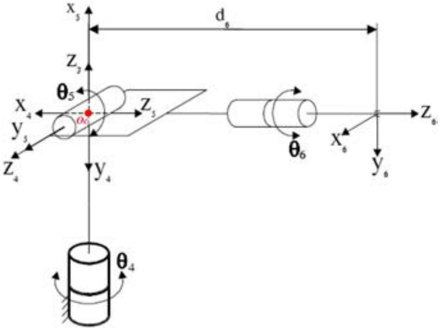
$$R_6^0(q_0, \cdots, q_n) = R$$
$$O_6^0(q_0, \cdots, q_n) = O$$

- Use the position of the wrist center to determine the first three joint angles.

Kinematic Decoupling

Spherical wrist as paradigm





- O_c is the intersection of the last 3 joint axes (z_3, z_4, and z_5);
- origins O_4 and O_5 will always be at O_c;
- motion of joints 4, 5 and 6 will **not** change the position of O_c;
- only motions of joints 1, 2 and 3 can influence position of O_c.

link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4
5	0	90	0	θ_5
6	0	0	d_6	θ_6

Kinematic Decoupling

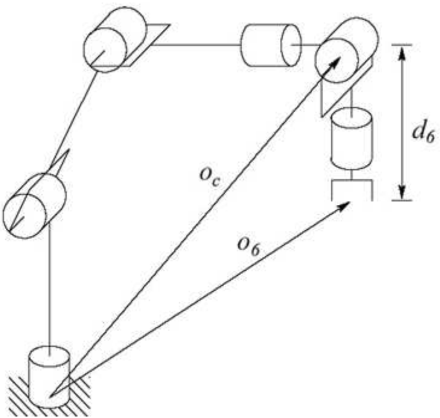
- Now, origin of tool frame, O_6, is a distance d_6 translated along z_5 (since z_5 and z_6 are collinear)
  - Thus, the third column of R is the direction of z_6 (w.r.t base frame) and we can write:
    $O = O_6^0 = O_c^0 + d_6 R \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$

- Rearranging: $O_c^0 = O - d_6 R \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$
- Recalling:

$$O = \begin{bmatrix} O_x & O_y & O_z \end{bmatrix}^T,$$
$$O_c^0 = \begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^T$$

- Then:

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} O_x - d_6\, r_{13} \\ O_y - d_6\, r_{23} \\ O_z - d_6\, r_{33} \end{bmatrix} \Rightarrow \theta_1, \theta_2, \theta_3$$



Kinematic Decoupling

- Since $\begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^T$ are determined from the first three joint angles,
  - our forward kinematics expression now allows to solve for the first 3 joint angles **decoupled** from the final 3.
  - Thus **we now have $\mathbf{R_3^0}$**
- Note that: $R = R_3^0 R_6^3$
- To solve for the final three joint angles:

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R \Rightarrow \theta_4, \theta_5, \theta_6$$

- Since the last three joints for a spherical wrist, we can use a set of Euler angles to solve for them

Inverse Position

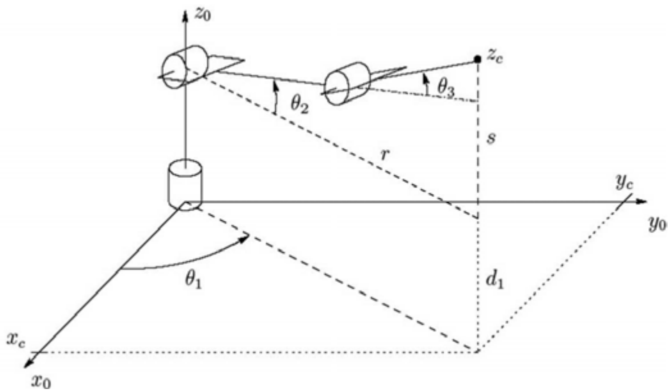
Now that we have $\begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^T$ we need to find $\theta_1, \theta_2, \theta_3$

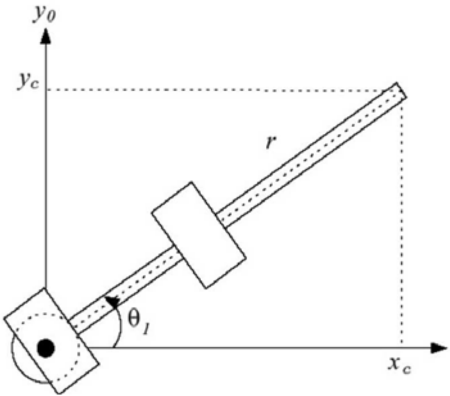
- Solve for θ_i by projecting onto the $\{x_{i-1}, y_{i-1}\}$ plane, solve trig problem
- Two examples:
  - elbow (RRR) and
  - spherical (RRP) manipulators

Inverse Position

Example: RRR Manipulator

- to solve for θ_1, project the arm onto the $\{x_0, y_0\}$ plane: $\theta_1 = \text{atan2}(y_c, x_c)$
- We can also have: $\theta_1 = \pi + \text{atan2}(y_c, x_c)$
- This will of course change the solutions for θ_2 and θ_3

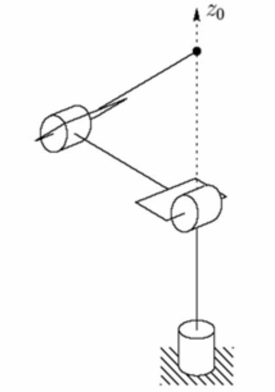




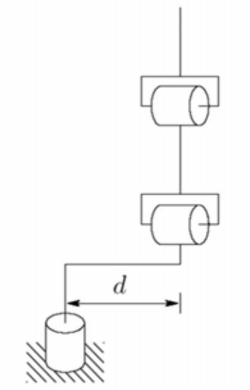
Inverse Position

Singular Configurations, Offsets

- If $x_c = y_c = 0$, θ_1 is undefined
- i.e. any value of θ_1 will work

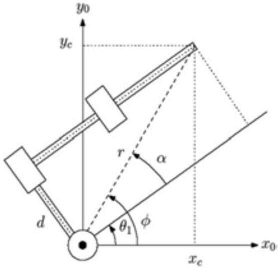


- If there is an **offset**, then we will have two solutions for θ_1: **left arm** and **right arm**
- However, wrist centers **can not** intersect z_0



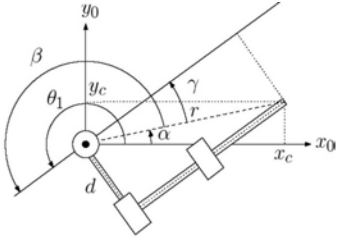
- **Left Arm**

$$\theta_1 = \phi - \alpha$$
$$\phi = \mathrm{atan2}(y_c, x_c)$$
$$\alpha = \mathrm{atan2}\left(d, \sqrt{x_c^2 + y_c^2 - d^2}\right)$$



- **Right Arm**

$$\theta_1 = \alpha + \beta$$
$$\alpha = \mathrm{atan2}(y_c, x_c)$$
$$\beta = \pi + \mathrm{atan2}\left(d, \sqrt{x_c^2 + y_c^2 - d^2}\right)$$
$$= \mathrm{atan2}\left(-d, -\sqrt{x_c^2 + y_c^2 - d^2}\right)$$



Inverse Position

Left and Right Arm Solutions

- Therefore there are in general two solutions for θ_1
- Finding θ_2 and θ_3 is identical to the planar two-link manipulator we have seen previously:

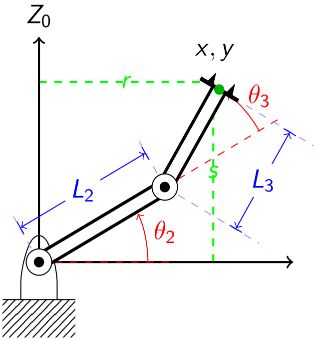
$$\cos\theta_3 = \frac{r^2 + s^2 - L_2^2 - L_3^2}{2L_2L_3}$$

$$r^2 = x_c^2 + y_c^2 - d^2$$

$$s = z_c - d_1$$

$$\cos\theta_3 = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - L_2^2 - L_3^2}{2L_2L_3}$$

$$\equiv D$$



- Therefore we can find two solutions for θ_3
  $\theta_3 = \text{atan2}\left(D, \pm\sqrt{1 - D^2}\right)$

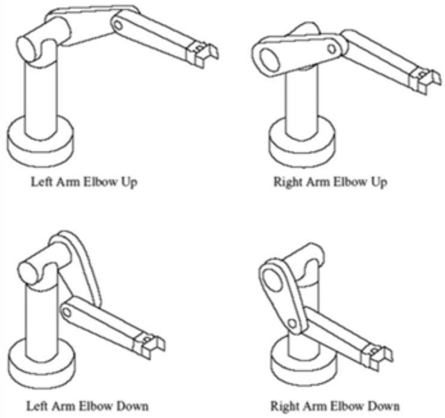
- The two solutions for θ_3 correspond to the elbow-down and elbow-up positions respectively
- Now solve for θ_2:

$$\begin{aligned}\theta_2 &= \text{atan2}\left(r, s\right) - \text{atan2}\left(L_2 + L_3 c_3, L_3 s_3\right) \\ &= \text{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) - \text{atan2}\left(L_2 + L_3 c_3, L_3 s_3\right)\end{aligned}$$

- Thus there are **two** solutions for the pair (θ_2, θ_3)

RRR: Four total solutions

- In general, there will be a maximum of **four solutions** to the inverse position kinematics of an elbow manipulator
- The 6R PUMA arm (as an example of the articulated geometry):

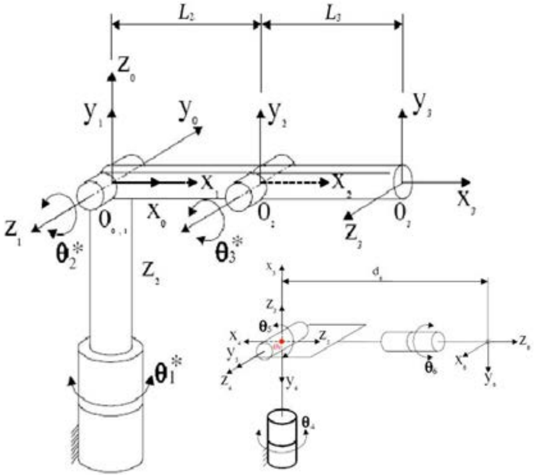


Inverse Orientation Problem

link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1
2	L_2	0	0	θ_2
3	L_3	0	0	θ_3
4	0	-90	0	θ_4
5	0	0	0	θ_5
6	0	0	d_6	θ_6

$$R_3^0 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 \\ s_1 c_{23} & -s_1 s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}$$

$$R_6^3 = \begin{bmatrix} (c_4 c_5 c_6 - s_4 s_6) & (-c_4 c_5 s_6 - s_4 c_6) & c_4 s_5 \\ (s_4 c_5 c_6 + c_4 s_6) & (-s_4 c_5 s_6 + c_4 c_6) & s_4 s_5 \\ -s_5 c_6 & s_5 c_6 & c_5 \end{bmatrix}$$



- Equation to solve: $R_6^3 = (R_3^0)^T R$

Inverse Orientation Problem

- Euler angle solutions can be applied. Taking the third column of $(R_3^0)^T R$

$$
\begin{aligned}
c_4 s_5 &= c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33} \\
s_4 s_5 &= -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33} \\
c_5 &= s_1 r_{13} - c_1 r_{23}
\end{aligned}
$$

- Again, if $\theta_5 \neq 0$, we can solve for θ_5:

$$
\theta_5 = \operatorname{atan2}\left(s_1 r_{13} - c_1 r_{23}, \pm\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}\right)
$$

- Finally, we can solve for the two remaining angles as follows:

$$
\begin{aligned}
\theta_4 &= \operatorname{atan2}(c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}, -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}) \\
\theta_6 &= \operatorname{atan2}(-s_1 r_{11} + c_1 r_{21}, s_1 r_{12} - c_1 r_{22})
\end{aligned}
$$

- For the singular configuration ($\theta_5 = 0$), we can only find $\theta_4 + \theta_6$ thus it is common to arbitrarily set θ_4 and solve for θ_6

Thanks for your attention.

Questions?

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