Robotics

CSE4316

Assoc. Prof. Dr.Ing.

Mohammed Nour

mnahmed@eng.zu.edu.eg

https://mnourgwad.github.io/CSE4316

Lecture 7: Robot Inverse Kinematics

Zagazig University | Faculty of Engineering | Computer and Systems Engineering Department | Zagazig, Egypt

Lecture: 7

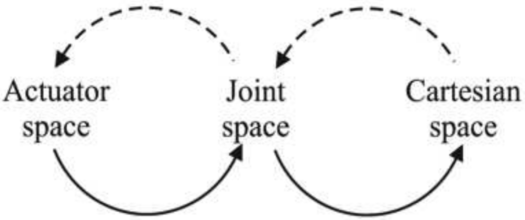
Robot Inverse Kinematics

- Inverse Kinematics
  - Geometric Approach
  - Algebraic Approach

Robot Inverse Kinematics

Three representations of manipulator position and orientation:

1. actuator space
2. joint space
3. Cartesian space



- In Forward kinematics we converted from **Joint space** to **Cartesian Space**
- To reach a specific point in 3D space we need to get the required joint angles

Robot Inverse Kinematics

Two Approaches

Geometric Approach

- we try to decompose the spatial geometry of the arm into several plane-geometry problems.
- For many manipulators (particularly when $\alpha = 0^\circ$ or $\pm 90^\circ$) this can be done quite easily.

Algebraic Approach

- We use link parameters to find the kinematic equations.
- the resulting transformation matrix is equated to goal specifications, and further manipulated to get the joint angles.

Simple Example

Example

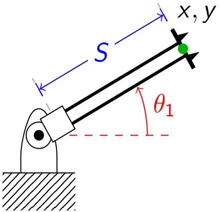
For this simple RP robot, find its inverse kinematics (i.e. find θ_1, S as a function of X and Y)

Finding S $\quad S = \sqrt{x^2 + y^2}$

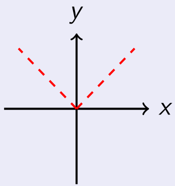
Finding θ_1 $\quad \theta_1 = \arctan\left(\frac{y}{x}\right)$

More Specifically

$$\theta_1 = \arctan 2\left(\frac{y}{x}\right)$$



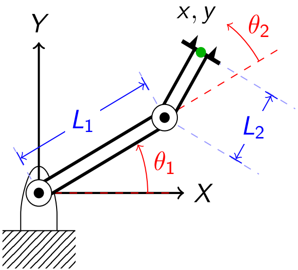
$$arctan2(y,x) = \begin{cases} \arctan(y/x), & x > 0 \\ \arctan(y/x) + \pi, & x < 0, \quad y \geq 0 \\ \arctan(y/x) - \pi, & x < 0, \quad y < 0 \\ +\pi, & x = 0, \quad y > 0 \\ -\pi, & x = 0, \quad y < 0 \\ \text{undifined}, & x = 0, \quad y = 0 \end{cases}$$



Two Link Manipulator

Example

For this RR robot, find its inverse kinematics (i.e.given: x, y and L_1, L_2; find: θ_1, θ_2)



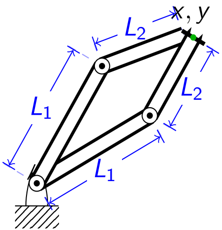
Existence of solutions

- For a solution to exist, the specified goal point must lie within the workspace.
- no configuration *singularity*

Redundancy:
no unique solution exists but we have **multiple solutions**.

Notice that for this manipulator:

- two solutions are possible.
- Sometimes no solution is possible.



Two Link Manipulator

Geometric Solution

Using the Law of Sines:

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin \bar{\theta}_1}{L_2} = \frac{\sin(180 - \theta_2)}{\sqrt{x^2 + y^2}} = \frac{\sin(\theta_2)}{\sqrt{x^2 + y^2}}$$

$$\theta_1 = \alpha - \bar{\theta}_1, \quad \alpha = \arctan 2(\frac{y}{x})$$

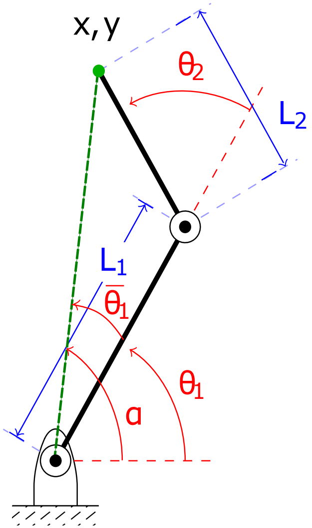
Using the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$(x^2 + y^2) = L_1^2 + L_2^2 - 2L_1L_2\cos(180 - \theta_2)$$

$$\cos(180 - \theta_2) = -\cos(\theta_2)$$

$$\cos(\theta_2) = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2}$$



Two Link Manipulator

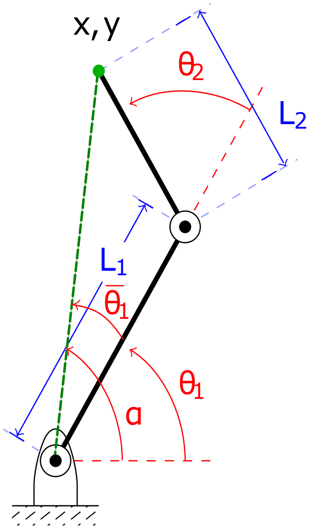
Geometric Solution

$$\theta_2 = \arccos\left(\frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2}\right)$$

Redundant since θ_2 could be in the 1^{st} or 4^{th} quadrant.

$$\theta_1 = \arctan 2\left(\frac{y}{x}\right) - \arcsin\left(\frac{L_2 \sin(\theta_2)}{\sqrt{x^2 + y^2}}\right)$$

Redundancy, since θ_2 has two possible values



We can see that:

$$x = L_1 c_1 + L_2 c_{1+2} \quad (1)$$

$$y = L_1 s_1 + L_2 s_{1+2} \quad (2)$$

squaring and summing 1 and 2, we get:

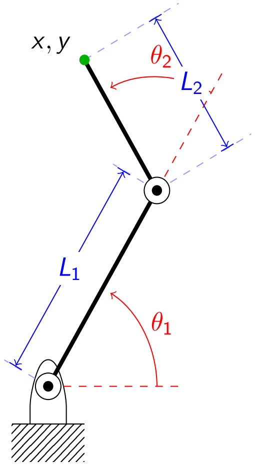
$$\begin{aligned} x^2 + y^2 &= L_1^2 c_1^2 + L_2^2 c_{1+2}^2 + 2L_1 L_2 c_1 c_{1+2} \\ &+ L_1^2 s_1^2 + L_2^2 s_{1+2}^2 + 2L_1 L_2 s_1 s_{1+2} \\ &= L_1^2 + L_2^2 + 2L_1 L_2 \left[c_1 c_{1+2} + s_1 s_{1+2}\right] \\ &= L_1^2 + L_2^2 + 2L_1 L_2 C_2 \end{aligned}$$

here C_2 is the only unknown

Remember

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$



Two Link Manipulator

Algebraic Solution

We finally get:

$$\theta_2 = \arccos\left(\frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2}\right)$$

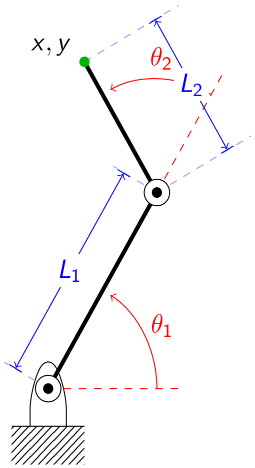
Rearranging 1 and 2, we get:

$$\begin{aligned} x &= L_1c_1 + L_2c_{1+2} \\ &= L_1c_1 + L_2c_1C_2 - L_2s_1s_2 \\ &= c_1(L_1 + L_2C_2) - s_1(L_2s_2) \end{aligned} \tag{3}$$

following the same procedure with 2:

$$y = c_1(L_2s_2) + s_1(L_1 + L_2C_2) \tag{4}$$

θ_2 is known so we can solve for θ_1.



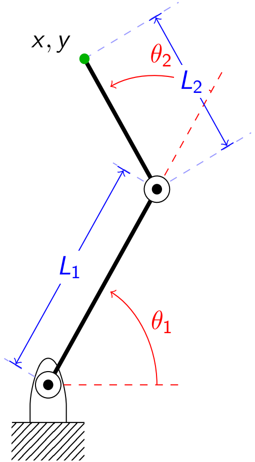
In equations 3 and 4, we have 2 equations and 2 unknowns (s_1 and c_1).
They can be rearranged as:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (L_1 + L_2 C_2) & -(L_2 s_2) \\ (L_2 s_2) & (L_1 + L_2 C_2) \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}$$

Utilizing the law of cosines to replace with $x^2 + y^2$ then substituting for c_1 and simplifying many times, we get:

$$s_1 = \frac{y(L_1 + L_2 C_2) - x L_2 s_2}{x^2 + y^2}$$

$$\theta_1 = \arcsin\left(\frac{y(L_1 + L_2 C_2) - x L_2 s_2}{x^2 + y^2}\right)$$



Algebraic Solution

Find the values of joint parameters that will put the tool frame at a desired position and orientation (within the workspace)

- Given the transformation matrix: $H = \begin{bmatrix} R_{3\times3} & T_{3\times1} \\ 0 & 1 \end{bmatrix}, R \in SO(3)$
- Find **all** solutions to: $T_n^0(q_1, \cdots, q_n) = H$
- Noting that: $T_n^0(q_1, \cdots, q_n) = A_1(q_1) \cdots A_n(q_n)$
- This gives **12** (nontrivial) equations with n unknowns
- with end effector position at $T_{3\times1}$
- and its orientation is obtained as:

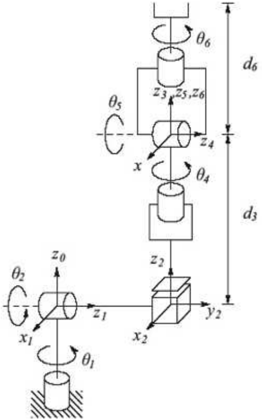
$$\psi = \text{atan2}\left(\frac{R_{21}}{R_{11}}\right), \quad \theta = \text{atan2}\left(-\frac{R_{31}\sin(\psi)}{R_{21}}\right), \quad \phi = \text{atan2}\left(\frac{R_{32}}{R_{33}}\right).$$

Robot Inverse Kinematics

Example: the Stanford manipulator

DH Parameters:

link	a_i	α_i	d_i	θ_i
1	0	-90	0	θ_1
2	0	90	d_2	θ_2
3	0	0	d_3	0
4	0	-90	0	θ_4
5	0	90	0	θ_5
6	0	0	d_6	θ_6



$d_2 = 0.154, d_6 = 0.263$

From which we get the transformation matrix (in general form):

$$H_6^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = [r_{ij}], \quad \begin{matrix} i = 1, \cdots, 4, \\ j = 1, \cdots, 4 \end{matrix}$$

Robot Inverse Kinematics

- For a given:

$$H_6^0 = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Find $\mathbf{q} = [\theta_1, \theta_2, d3, \theta_4, \theta_5, \theta_6]^T$.

- We have 12 **non-trivial** equations:

$$\begin{aligned}
c_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] - d_2\left(s_4c_5c_6 + c_4s_6\right) &= 0 \\
s_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] + c_1\left(s_4c_5c_6 + c_4s_6\right) &= 0 \\
-s_2\left(c_4c_5c_6 - s_4s_6\right) - c_2s_5c_6 &= 1 \\
c_1\left[-c_2\left(c_4c_5s_6 + s_4c_6\right) + s_2s_5s_6\right] - s_1\left(-s_4c_5s_6 + c_4c_6\right) &= 1 \\
-s_1\left[-c_2\left(c_4c_5s_6 - s_4c_6\right) - s_2s_5s_6\right] + c_1\left(-s_4c_5s_6 + c_4s_6\right) &= 0 \\
s_2\left(c_4c_5s_6 + s_4c_6\right) + c_2s_5s_6 &= 0 \\
c_1\left(c_2c_4s_5 + s_2c_5\right) - s_1s_4s_5 &= 0 \\
s_1\left(c_2c_4s_5 + s_2c_5\right) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6\left(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5\right) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6\left(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2\right) &= 0.763 \\
c_2d_3 + d_6\left(c_2c_5 - c_4s_2s_5\right) &= 0
\end{aligned}$$

- One solution: $\mathbf{q} = [\pi/2, \pi/2, 0.5, \pi/2, 0, \pi/2]^T$.
- next lecture, we will see how to systematically find such solutions.

Thanks for your attention.

Questions?

Assoc. Prof. Dr.Ing.
Mohammed Nour
mnahmed@eng.zu.edu.eg
https://mnourgwad.github.io/CSE4316

Robotics Research Interest Group (zuR^2IG)
Zagazig University | Faculty of Engineering | Computer and Systems Engineering Department | Zagazig, Egypt