Robotics
CSE4316

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Lecture 5: Robot Kinematics (cont.)

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Lecture 5

Robot Kinematics (cont.)

- Homogeneous Transformations,
- Robot Kinematics
- Forward Kinematics

3D Rotations

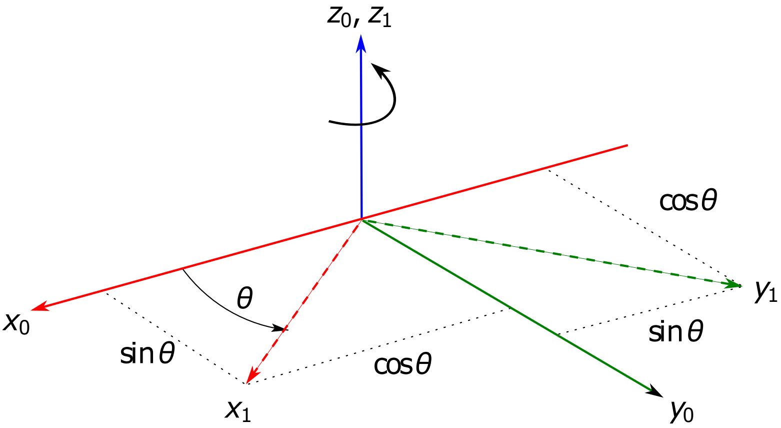
Rotation about z_0 by an angle θ

$$x_1^0 = \begin{bmatrix} \vec{x_1} \cdot \vec{x_0} \\ \vec{x_1} \cdot \vec{y_0} \\ \vec{x_1} \cdot \vec{z_0} \end{bmatrix},$$

$$y_1^0 = \begin{bmatrix} \vec{y_1} \cdot \vec{x_0} \\ \vec{y_1} \cdot \vec{y_0} \\ \vec{y_1} \cdot \vec{z_0} \end{bmatrix},$$

$$z_1^0 = \begin{bmatrix} \vec{z_1} \cdot \vec{x_0} \\ \vec{z_1} \cdot \vec{y_0} \\ \vec{z_1} \cdot \vec{z_0} \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

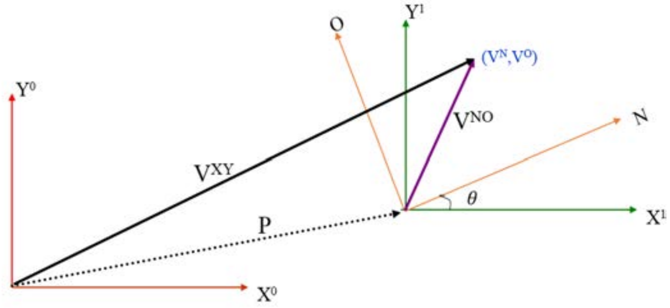


Frame Transformation

Translation Followed by Rotation

- **Translation** along P followed by **rotation** by θ around Z-axis

$$v^{XY} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_N \\ v_O \end{bmatrix}$$



- P_x, P_y are relative to the original coordinate frame.
- Translation followed by rotation is different than rotation followed by translation.
- knowing the coordinates of a point (V_N, V_O) in some coordinate frame (NO) you can find the position of that point relative to another coordinate frame (X_0, Y_0).

Homogeneous representation

Putting It All into a Matrix

- for a **general** frame transformation (translation followed by rotation)

$$v^{XY} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}_{2\times1} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{2\times2} \begin{bmatrix} v_N \\ v_O \end{bmatrix}_{2\times1}$$

- Padding with 0's and 1's

$$v^{XY} = \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_N \\ v_O \\ 1 \end{bmatrix}$$

- Simplifying into a matrix form

$$v^{XY} = \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_x \\ \sin\theta & \cos\theta & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_N \\ v_O \\ 1 \end{bmatrix}$$

- **Homogenous Matrix** for a translation in XY–plane, followed by a rotation around the z-axis

Homogeneous representation

- Transformation matrices has the general form:

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}; \quad R \in SO(3), d \in \mathbb{R}^3$$

- it is called homogeneous transformations.
- it represents a rigid motion.
- using the fact that R is orthogonal, the inverse of transformation H is given by:

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} \qquad \textbf{[Prove it !]}$$

- H is a special case of homogeneous coordinates, (extensively used in field of computer graphics):

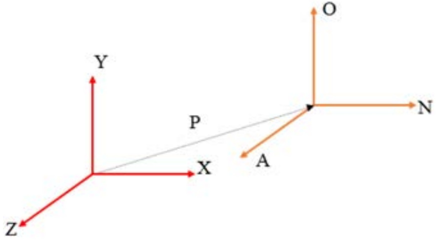
$$H = \left[\begin{array}{c|c} R_{3\times3} & d_{3\times1} \\ \hline f_{1\times3} & s_{1\times1} \end{array}\right] = \left[\begin{array}{c|c} \texttt{Rotation} & \texttt{Translation} \\ \hline \texttt{perspective} & \texttt{scale factor} \end{array}\right]$$

Homogeneous Matrices in 3D

H is a 4×4 matrix that can describe a translation, rotation, or both in **one matrix**

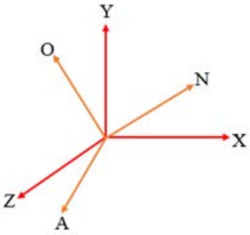
- **Translation without rotation**

$$H = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- **Rotation without translation**

$$H = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- Rotation part: Could be rotation around z–, x–, y–axis, or a combination of the three.

Homogeneous Matrices in 3D

Finding the Homogeneous Matrix

- The (n, o, a) position of a point relative to the current coordinate frame you are in.

$$v^{XY} = H \begin{bmatrix} v^N \\ v^O \\ v^A \\ 1 \end{bmatrix}$$

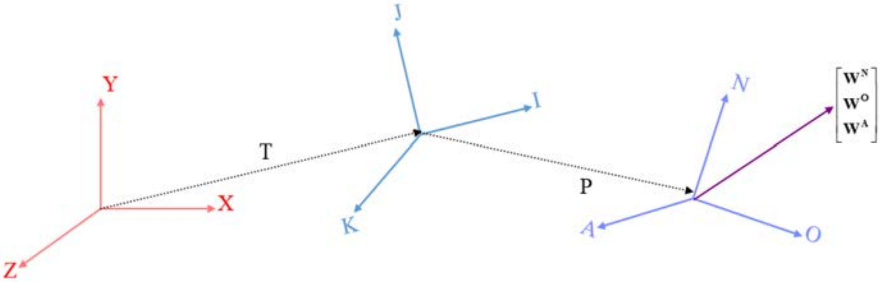
$$= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v^N \\ v^O \\ v^A \\ 1 \end{bmatrix}$$

$$V^X = n_x V^N + o_x V^O + a_x V^A$$

- The rotation and translation parts can be combined into a single homogeneous matrix IIF both are **relative to the same coordinate frame**.

Homogeneous Transformation

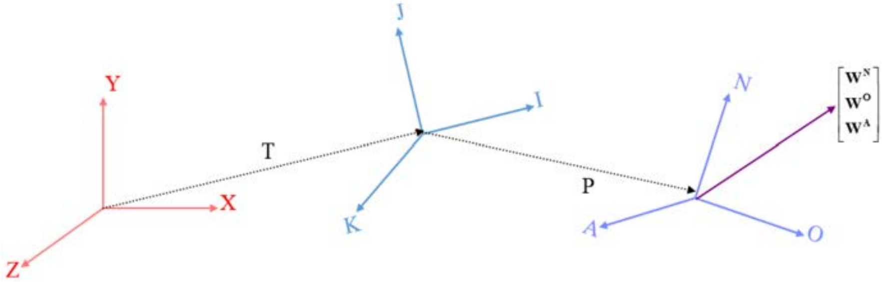
Example



$$\begin{bmatrix} W^X \\ W^Y \\ W^Z \\ 1 \end{bmatrix} = \begin{bmatrix} i_x & j_x & k_x & T_x \\ i_y & j_y & k_y & T_y \\ i_z & j_z & k_z & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_i & o_i & a_i & P_i \\ n_j & o_j & a_j & P_j \\ n_k & o_k & a_k & P_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} W^N \\ W^O \\ W^A \\ 1 \end{bmatrix}$$

Homogeneous Transformation

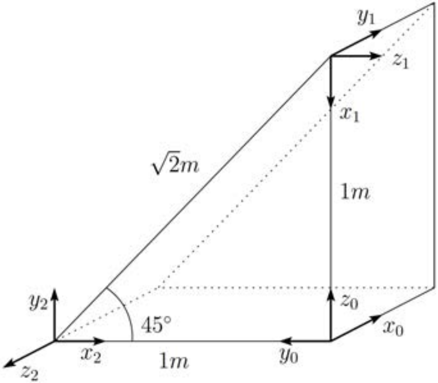
Example



- The Homogeneous Matrix is a concatenation of numerous translations and rotations
- for the previous example:

$$H = (\text{Translation relative to the XYZ frame}) * (\text{Rotation relative to the XYZ frame})$$
$$* (\text{Translation relative to the IJK frame}) * (\text{Rotation relative to the IJK frame})$$

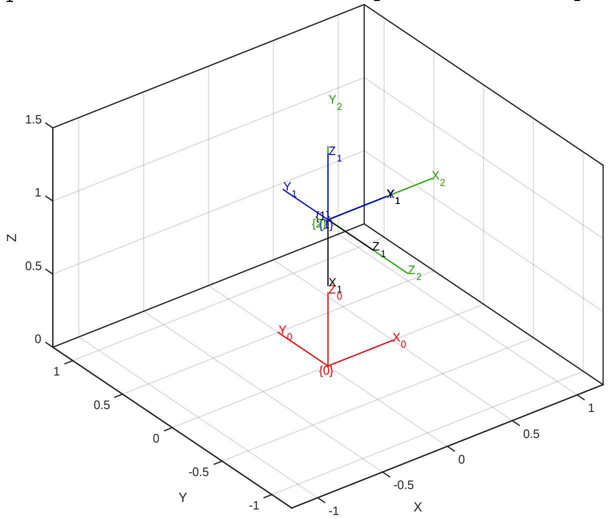
- Find the homogeneous transformations:
  H_1^0, H_2^0, H_2^1
  representing transformations among the 3 frames.
- Show that: $H_2^0 = H_1^0 \, H_2^1$.

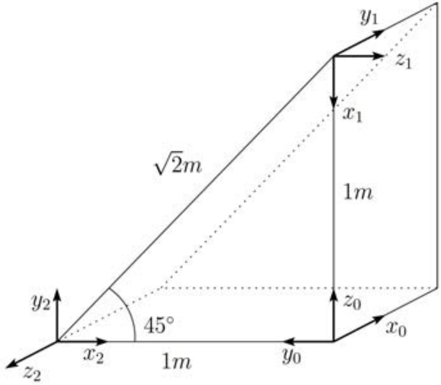


Homogeneous Transformation

Full Example

$H_1^0 = \texttt{transl(0, 0, 1) * trotx(pi/2) * trotz(-pi/2)}$





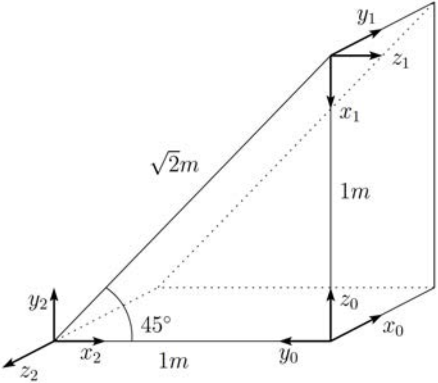
Homogeneous Transformation

Full Example

$$H_1^0 = \texttt{transl(0, 0, 1) * trotx(pi/2) * trotz(-pi/2)}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



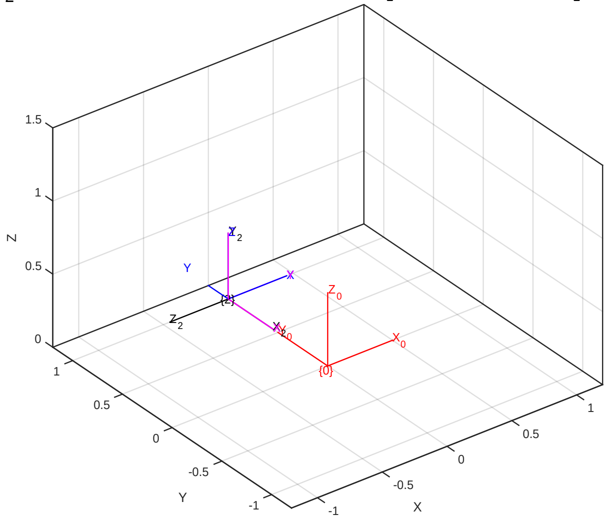
$$R_Z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_Y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

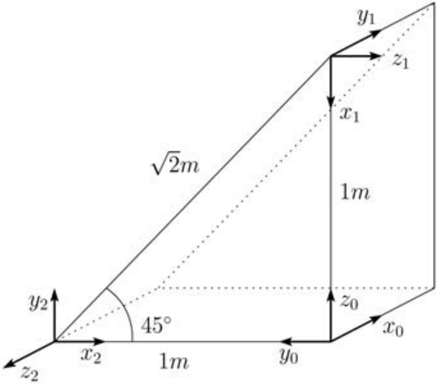
$\cos(\pm 90) = \sin(0) = 0,$
$cos(0) = \sin(\pm 90) = \pm 1$

Homogeneous Transformation

Full Example

$H_2^0 =$ `transl(0, 1, 0) * trotz(-pi/2) * trotx(pi/2)`





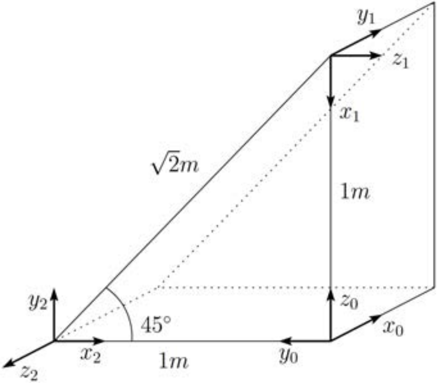
Homogeneous Transformation

Full Example

$$H_2^0 = \texttt{transl(0, 1, 0) * trotz(-pi/2) * trotx(pi/2)}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



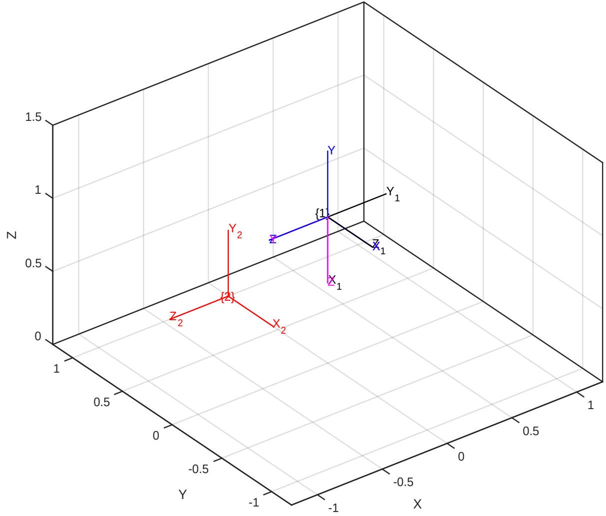
$$R_Z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_Y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

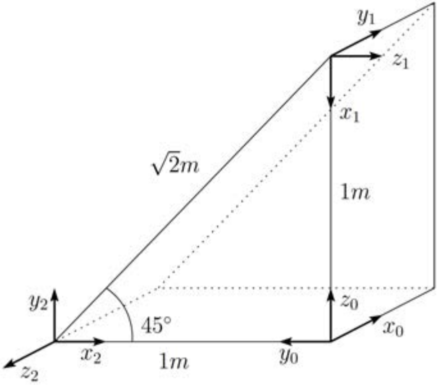
$\cos(\pm 90) = \sin(0) = 0,$
$cos(0) = \sin(\pm 90) = \pm 1$

Homogeneous Transformation

Full Example

$H_2^1 =$ `transl(0, 1, 0) * trotz(-pi/2) * trotx(pi/2)`





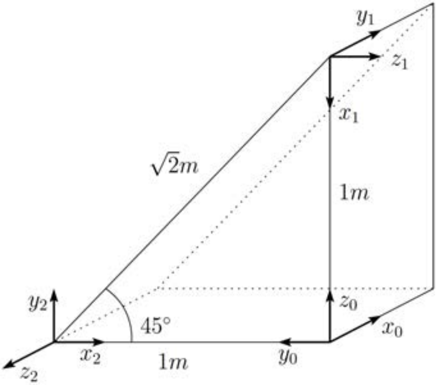
Homogeneous Transformation

Full Example

$H_2^1 = \texttt{transl(1, 1, 0) * trotx(-pi/2) * troty(pi/2)}$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$R_Z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_Y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

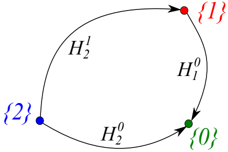
$\cos(\pm 90) = \sin(0) = 0,$
$cos(0) = \sin(\pm 90) = \pm 1$

Homogeneous Transformation

Full Example

$$H_1^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_2^0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$H_2^1 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$H_1^0 H_2^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = H_2^0$$

Homogeneous Transformation

Rotation matrix, either $R \in SO(3)$ or $R \in SO(2)$, can be interpreted in 3 distinct ways, it :

1. represents a coordinate transformation relating the coordinates of a point p in two different frames.
2. gives the orientation of a transformed coordinate frame w.r.t a fixed coordinate frame.
3. is an operator taking a vector and rotating it to a new vector in the same coordinate system.

Homogeneous transformations are **used** as:

1. coordinate transforms
   - on position vectors
   - on free (velocity, etc.) vectors
2. relative position/orientation (or motion) between two frames
3. absolute motion in a fixed frame

Thanks for your attention.

Questions?

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