**Zagazig University,Faculty of Engineering** **Midterm Exam**
**Academic Year**: 2017/2018
**Specialization**: Computer & Systems Eng.
**Course Name**: Digital Control
**Course Code**: CSE421
**Examiner**: Dr. **Mohammed Nour**

**Date**: 11/11/2016
**Exam Time**: 60 min.
**No. of Pages**: 5
**No. of Questions**: 4 (20 items)
**Full Mark**: [60]

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▷ Please **answer all questions**. Use **3** decimal digits approximation.

▷ Mark your **answers** for all questions **in the Answer Sheet** provided.

▷ In last page, some supplementary identities you *may* need.

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## Question 1

[2 p] **1.Ballast Coding** is used to synchronize the sampling and control process. Which of the following **does not** apply for this method:

a) completely implemented in software.

b) software interrupt mechanism is used.

c) adds dummy code to compensate for required sampling interval.

d) very sensitive to changes in code and/or CPU clock rate.

**1.**Ballast Coding does **not** use any software (or hardware) interrupt mechanisms as it is fully implemented in software.

[2 p] **2.**For the two systems **A** and **B** represented by the following difference equations:

$$\mathbf{A}: y(k+2) = y(k+1)y(k) + u(k), \quad \mathbf{B}: y(k+5) = y(k+4) + u(k+1) - u(k),$$

a) **A** is homogeneous and **B** is linear.

b) **A** is time-variant and **B** is homogeneous.

c) both systems are linear and time-invariant.

d) both systems are time-invariant and homogeneous.

[4 p] **3.**The z-transform of the sequence: $\{0, 2^{-0.5}, 1, 2^{-0.5}, 0, 0, \cdots\}$ is: (note: $\sin(45°) = \frac{1}{\sqrt{2}}$)

a) $\dfrac{z^2 + z + 1}{\sqrt{2}z}$

b) $\dfrac{z^4 - 1}{z^3\left[z^2 - 2z + 1\right]}$

c) $\dfrac{z}{\left[\sqrt{2}z^2 - 2z + \sqrt{2}\right]}$

d) $\dfrac{z^4 - 1}{\sqrt{2}z^3\left[z^2 - \sqrt{2}z + 1\right]}$

**3.**From z-transform definition we can write:

$$\mathscr{Z}\{0, 2^{-0.5}, 1, 2^{-0.5}, 0, 0, \cdots\} = \frac{1}{\sqrt{2}z} + \frac{1}{z^2} + \frac{1}{\sqrt{2}z^3} = \frac{z^2 + \sqrt{2}z + 1}{\sqrt{2}z^3}$$

Since this answer is not given, then we try using z-transform proprieties to get:

$$\begin{aligned}\{0, 2^{-0.5}, 1, 2^{-0.5}, 0, 0, \cdots\} &= \{0, 2^{-0.5}, 1, 2^{-0.5}, 0, -2^{-0.5}, -1, -2^{-0.5}, 0, \cdots\} \\ &+ \{0, 0, 0, 0, 2^{-0.5}, 1, 2^{-0.5}, 0, -2^{-0.5}, -1, -2^{-0.5}, 0, \cdots\} \\ &= f(k) + g(k)\end{aligned}$$

where $f(k) = \begin{cases} \sin\left(\frac{k\pi}{4}\right), & k > 0 \\ 0, & k \le 0 \end{cases}$ $\quad g(k) = \begin{cases} \sin\left(\frac{k\pi}{4}\right), & k > 4 \\ 0, & k \le 4 \end{cases}$

$$\begin{aligned}\mathscr{Z}\{0, 2^{-0.5}, 1, 2^{-0.5}, 0, 0, \cdots\} &= \frac{\sin(\pi/4)z}{z^2 - 2\cos(\pi/4)z + 1} - z^{-4}\frac{\sin(\pi/4)z}{z^2 - 2\cos(\pi/4)z + 1} \\ &= \frac{2^{-0.5}\left(z^4 - 1\right)}{z^3\left[z^2 - 2^{0.5}z + 1\right]} = \frac{z^4 - 1}{\sqrt{2}\, z^3\left[z^2 - \sqrt{2}\, z + 1\right]}\end{aligned}$$

[4 p] **4.** The inverse transform of the function: $F(z) = \dfrac{z}{z^2 + 0.3\,z + 0.02}$ is:

a) $\{0, 1, -0.3, 0.07, \cdots\}$ b) $\{1, -0.3, 0.07, \cdots\}$ c) $\{0, 1, 0.3, 0.02, \cdots\}$ d) $\{1, 0.3, 0.02, \cdots\}$

**4.** since the answer is required in expansion form, we use the long division method as:

$$\begin{array}{r|l} & z^{-1} - 0.3z^{-2} + 0.07z^{-3} + \cdots \\ z^2 + 0.3z + 0.02 & z \\ & \underline{z + 0.3 + 0.02z^{-1}} \\ & -0.3 - 0.02z^{-1} \\ & \underline{-0.3 - 0.09z^{-1} - 0.006z^{-2}} \\ & 0.07z^{-1} + 0.006z^{-2} \\ & \vdots \end{array}$$

from which we have:

$$F(z) = \frac{z}{z^2 + 0.3z + 0.02} = z^{-1} - 0.3z^{-2} + 0.07z^{-3} + \cdots \quad \Rightarrow \quad f(k) = \{0, 1, -0.3, 0.07, \cdots\}$$

[3 p] **5.** The inverse transforms of the function: $F(z) = \dfrac{z - 0.1}{z^2 + 0.04z + 0.25}$ is:

a) $\sin(1.611k + 0.196)$

b) $\cos(1.611k + 0.196)$

c) $-0.4\delta(k) + 2.057(0.5)^k \sin(1.611k + 0.196)$

d) $-0.4\delta(k) + \cos(1.611k + 0.196)$

**5.** since the answer is required in closed form, we use the partial fraction method as:

$$\begin{aligned}\frac{F(z)}{z} &= \frac{z - 0.1}{z\left(z^2 + 0.04z + 0.25\right)} = -\frac{0.4}{z} + \frac{0.4z + 1.016}{z^2 + 0.04z + 0.25} \\ &= -0.4 + \frac{0.4z^2 + 1.016z}{z^2 + 0.04z + 0.25}\end{aligned}$$

From the z-transform table, we can use the identities:

$$\mathscr{Z}\Big\{e^{-\alpha k}\sin(k\omega_d)\Big\} = \frac{e^{-\alpha}\sin(\omega_d)z}{z^2 - 2e^{-\alpha}\cos(\omega_d)z + e^{-2\alpha}}$$

$$\mathscr{Z}\Big\{e^{-\alpha k}\cos(k\omega_d)\Big\} = \frac{z[z - e^{-\alpha}\cos(\omega_d)]}{z^2 - 2e^{-\alpha}\cos(\omega_d)z + e^{-2\alpha}}$$

$$e^{-\alpha} = \sqrt{0.25} = 0.5, \quad \cos(\omega_d) = -0.04 \Rightarrow \omega_d = 1.61 \text{ rad}$$

$$F(z) = \frac{0.4z^2 + 1.016z}{z^2 + 0.04z + 0.25} = \frac{0.4\left(z^2 + 0.02z\right) + 1.008z}{z^2 + 0.04z + 0.25} = \frac{0.4\left(z^2 + 0.02z\right) + 2.018\,(0.4996)\,z}{z^2 + 0.04z + 0.25}$$

$$\mathscr{Z}^{-1}\{F(z)\} = \{\text{f}(k)\} = -0.4\delta(k) + (0.5)^k\,[0.4\cos(1.611k) + 2.018\sin(1.611k)]$$

$$= -0.4\delta(k) + 2.057(0.5)^k\sin(1.611k + 0.196)$$

[3 p] **6.** If the discrete output $y_k$ of a system is related to its input $u_k$ by $y_n = \sum_{k=0}^{n} u_k$, the transfer function $Y(z)/U(z)$ of this system is:

a) $\dfrac{1}{1 - z^{-1}}$  b) $\dfrac{z}{z^2 - 1}$  c) $1$  d) $z^{-1}$

**6.** We can write:

$$y_n = \sum_{k=0}^{n} u_k \quad \Rightarrow \quad y_{n-1} = \sum_{k=0}^{n-1} u_k$$

By subtracting, we obtain

$$y_n - y_{n-1} = u_n$$

Taking z-transform for both sides and equating, we get:

$$\frac{Y/(z)}{U(z)} = \frac{1}{1 - z^{-1}}$$

[3 p] **7.** Consider a causal, LTI system with $x_n$ and $y_n$ as input and output, resp., described by the difference equation $y_n = a\ y_{n-1} + b\ x_n$. For which values of $a$ and $b$ is the system bounded-input bounded-output stable?

a) $a > 1$ & $b < 1$  b) $a < 1$ & $b > 1$.  c) $|b| < 1$ any $a$.  d) $|a| < 1$, any $b$

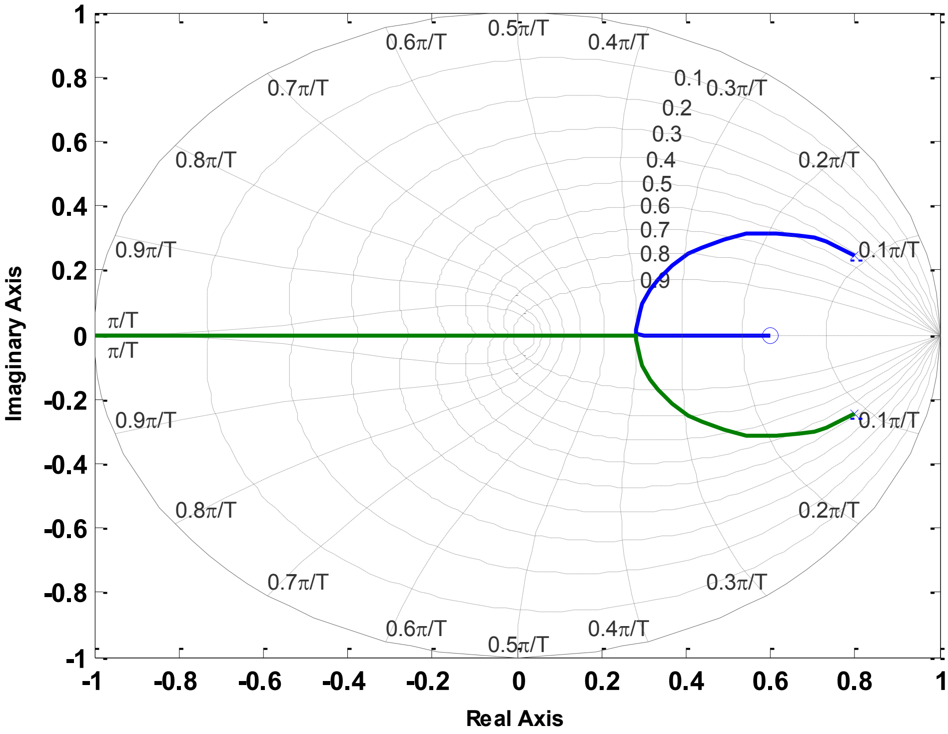
**7.** The transfer function of the system is given by:

$$H(z) = \frac{b}{1 - a\ z^{-1}}$$

which has a pole at $z = a$. A causal LTI system is stable if and only if all the poles of $H(z)$ lie inside the unit circle. It follows that $|a| < 1$. Since $b$ is not related to any pole, so it does not affect the stability.

## Question 2

For the next questions, consider the next root Locus chart:




[3 p] **8.** The system represented by this root locus has a transfer function $G(z)$ given by:

a) $\dfrac{K(z^2 - 1.6\,z + 0.7)}{z - 0.6}$ b) $\dfrac{K(z - 0.6)}{z^2 - 1.6\,z + 0.7}$ c) $\dfrac{z - 0.4\pi/T}{(z^2 - 0.1\pi/T\,z + 0.6)}$ d) $\dfrac{(z^2 - 0.1\pi/T\,z + 0.6)}{z - 0.4\pi/T}$

**8.** From the root locus chart:

- open-loop Zeros are at 0.6
- open-loop Poles are at $0.8 \pm j\,0.25$

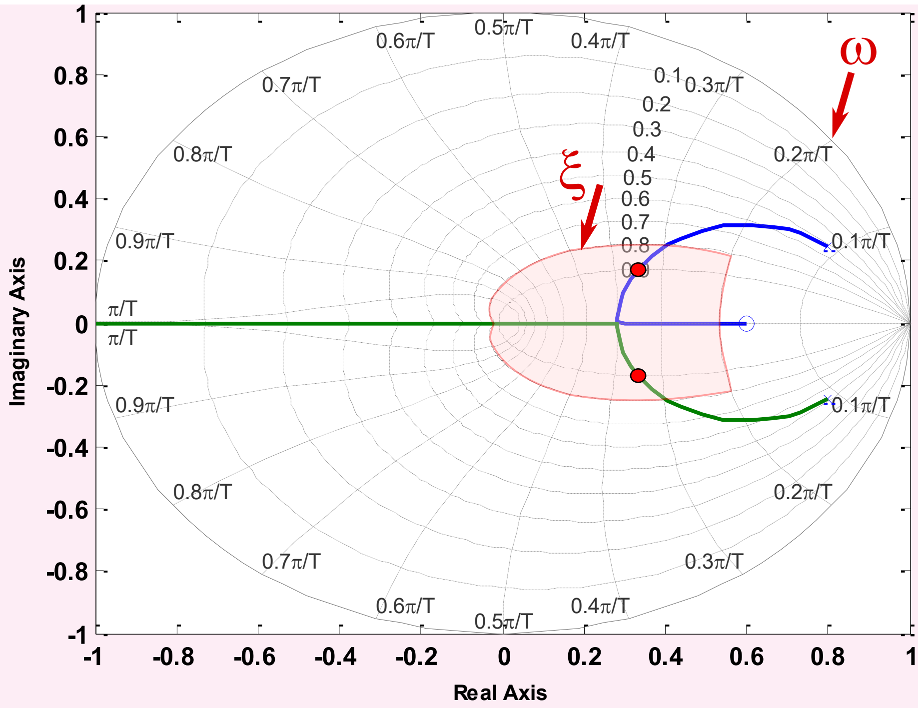
However, we cannot find the system gain, so leave it as $K$. Therefore:

$$G(z) = K\frac{z - 0.6}{(z - 0.8 - j\,0.25)(z - 0.8 + j\,0.25)} = K\frac{z - 0.6}{z^2 - 1.6\,z + 0.7}$$

[3 p] **9.** Using the given chart, the appropriate poles location that results in $\omega_n > 0.63$ rad/s , $\zeta > 0.8$ (assuming $T = 1$ s) are:

a) $0.33 \pm j\,0.17$ b) $0.17 \pm j\,0.33$ c) $0.6 \pm j\,0.2$ d) $0.2 \pm j\,0.6$

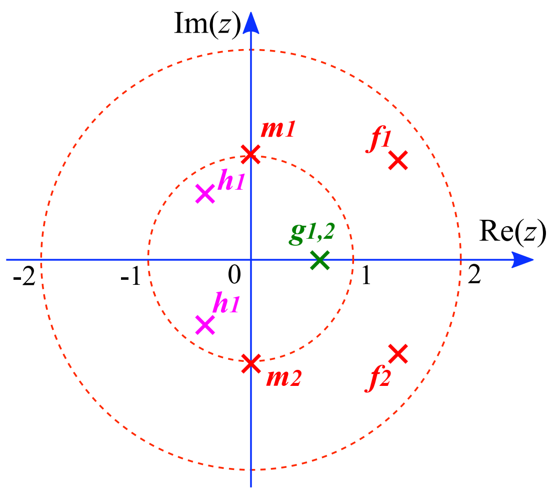
**9.** For $T = 1$ s, then $\omega_n > 0.63 \simeq 0.2\,\pi$
We can choose any poles that are within the **shaded area** and on the root locus:

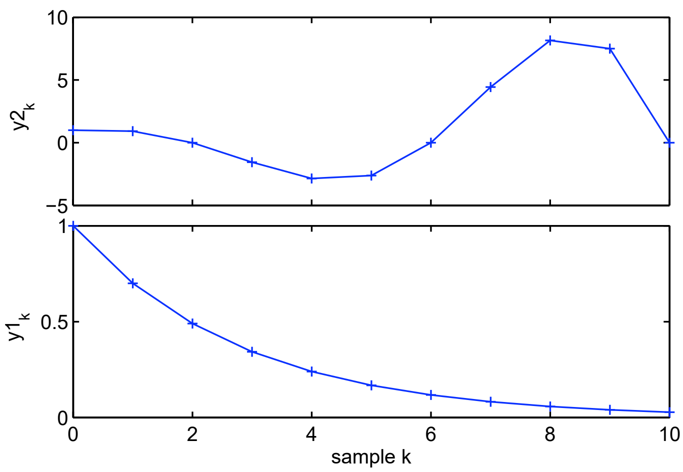



We select poles at $0.33 \pm j\,0.17$

## Question 2

The following diagrams represent systems with poles (indicated by x's) but no zeros. The unit step response of two systems $y1_k$ and $y2_k$ is shown on the right plots.




[3 p] **10.**The system with the response given by $y1_k$ has the poles:

a) $f_1, f_2$ b) $g_1, g_2$ c) $h_1, h_2$ d) $m_1, m_2$

**10.**

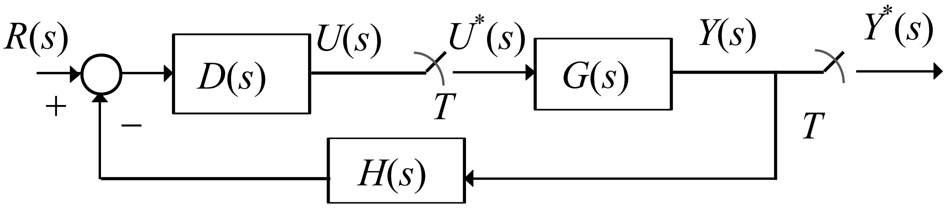
- The response $y1_k$ is **exponentially decreasing without oscillations**, so its roots must be all real and inside the unit circle. Therefore, its roots are $g_{1,2}$.

[3 p] **11.** The system with the response given by $y2_k$ has the poles:

a) $f_1, f_2$    b) $g_1, g_2$    c) $h_1, h_2$    d) $m_1, m_2$

**11.**
- The response $y2_k$ is exponentially **increasing with oscillations**, so its roots must be all complex conjugate and outside the unit circle. Therefore, its roots are $f_{1,2}$.

[3 p] **12.** In the following sampled system, $D(s), G(s)$, and $H(s)$ represent the system continuous subsystems. $R(s)$ and $Y(s)$ are input and output respectively:




The discrete response $Y(z)$ of this sampled system is:

a) $Y(z) = \dfrac{G(z)\,DR(z)}{1 + D(z)HG(z)}$    b) $Y(z) = \dfrac{G(z)\,D(z)\,R(z)}{1 + D(z)\,H(z)\,G(z)}$

c) $Y(z) = \dfrac{G(z)\,D(z)\,R(z)}{1 + DHG(z)}$    d) $Y(z) = \dfrac{G(z)\,DR(z)}{1 + DHG(z)}$

**12.** From the block diagram:

$$U(s) = D(s)\,R(s) - D(s)\,H(s)\,G(s)\,U^*(s)$$

Then sampling gives:

$$U^*(s) = [DR]^*(s) - [DHG]^*(s)\,U^*(s)$$

Solving for $U^*(s)$, we obtain:

$$U^*(s) = \frac{[DR]^*(s)}{1 + [DHG]^*(s)}$$

The analog output is:

$$Y(s) = G(s)\,U^*(s) = \frac{G(s)\,[DR]^*(s)}{1 + [DHG]^*(s)}$$

The discrete output $Y(z)$ is:

$$Y(z) = \frac{G(z)\,DR(z)}{1 + DHG(z)}$$

[3 p] **13.** The final value for the function $F(z) = \dfrac{z}{z^2 + 0.3\,z + 2}$ is:

a) 0    b) 3.3    c) 1/3.3    d) undefined

**13.** The denominator has complex conjugate poles with magnitude $\sqrt{2} > 1$. Therefore, the corresponding time sequence is unbounded and the final value theorem does not apply. hence, the final value for this function is **undefined**.

[3 p] **14.** The steady state DC gain of the system with transfer Function $H(z) = \dfrac{z}{z^2 - 0.7\,z + 0.1}$ is:

a) 0 b) 0.4 c) 2.5 d) undefined

**14.** The steady state DC gain of the system is calculated as:

$$K = H(1) = \left.\frac{z}{z^2 - 0.7\,z + 0.1}\right|_{z=1} = \frac{1}{1 - 0.7 + 0.1} = 2.5$$

[3 p] **15.** For a system with the following characteristic equation:

$$F(z) = z^5 - 0.25\,z^4 + 0.1\,z^3 + 0.4\,z^2 + 0.3\,z - 0.1$$

The Jury Table constructed to determine this system stability will have:

a) 5 rows b) 6 rows c) 7 rows d) 8 rows

**15.** The Jury table will have $2n - 3$ rows (always odd). Since $n = 5$, this characteristic equation will result in **7 rows**.

[3 p] **16.** The **filth** row of that Jury Table will be:

a) $-0.986 \quad 0.528 \quad 0.713 \quad -0.528$

b) $0.905 \quad 0.331 \quad 0.100 \quad 0.466$

c) $0.812 \quad -0.2752 \quad 0.2288$

d) $0.466 \quad 0.100 \quad 0.331 \quad 0.905$

**16.** The Jury table for this characteristic equation is constructed as:

| *Row* | $z^0$ | $z^1$ | $z^2$ | $z^3$ | $z^4$ | $z^5$ |
|---|---|---|---|---|---|---|
| 1 | −0.1 | 0.3 | 0.4 | 0.1 | −0.25 | 1 |
| 2 | 1 | −0.25 | 0.1 | 0.4 | 0.3 | −0.1 |
| 3 | −0.99 | 0.22 | −0.14 | −0.41 | −0.275 | |
| 4 | −0.275 | −0.41 | −0.14 | 0.22 | −0.99 | |
| 5 | 0.9045 | 0.3306 | 0.1001 | 0.4664 | | |
| 6 | 0.4664 | 0.1001 | 0.3306 | 0.9045 | | |
| 7 | 0.812 | −0.2752 | 0.2288 | | | |

## Question 3

In a unity feedback discrete system, its open loop transfer function is given as:

$$G(z) = \frac{k\left(0.084z^2 + 0.17z + 0.019\right)}{z^3 - 1.5z^2 + 0.553z - 0.05}$$

[3 p] **17.** The system characteristic equation is:

a) $G(z) = 0$

b) $G(z)\,H(z) = 0$

c) $1 + G(z) = 0$

d) $z^3 - 1.5z^2 + 0.553z - 0.05 = 0$

**17.** For this unity feedback system, its characteristic equation is:

$$P(z) = 1 + \frac{k(0.084z^2 + 0.17z + 0.019)}{(z^3 - 1.5z^2 + 0.553z - 0.05)} = 0$$
$$= z^3 + (0.084k - 1.5)z^2 + (0.17k + 0.553)z + (0.019k - 0.05) = 0$$

[3 p] **18.** The bilinear transformation used to transform the interior of the z-plane unit circle into the left-hand s-plane is:

a) $\frac{1-w}{1+w}$

b) $\frac{1+w}{1-w}$

c) $z = e^{sT}$

d) $z = e^{-sT}$

**18.** Transforming $P(z)$ into $w$-domain:

$$Q(w) = \left[\frac{w+1}{w-1}\right]^3 + (0.084k - 1.5)\left[\frac{w+1}{w-1}\right]^2 + (0.17k + 0.553)\left[\frac{w+1}{w-1}\right] + (0.019k - 0.05) = 0$$
$$= (0.003 + 0.27k)w^3 + (1.1 - 0.11k)w^2 + (3.8 - 0.27k)w + (3.1 + 0.07k) = 0$$

[3 p] **19.** The **third** row of the Routh–Hurwitz array will be:

a) $\frac{0.01\,k^2 - 1.55\,k + 4.17}{1.1 - 0.11\,k}$

b) $1.1 - 0.11\,k \qquad 3.1 + 0.07\,k$

c) $3.1 + 0.07\,k$

d) $0.003 + 0.27\,k \qquad 3.8 - 0.27\,k$

**19.** We can now construct the Routh–Hurwitz array as:

$$\begin{array}{c|cc} \omega^3 & 0.003 + 0.27\,k & 3.8 - 0.27\,k \\ \omega^2 & 1.1 - 0.11\,k & 3.1 + 0.07\,k \\ \omega^1 & \dfrac{0.01\,k^2 - 1.55\,k + 4.17}{1.1 - 0.11\,k} & \\ \omega^0 & 3.1 + 0.07\,k & \end{array}$$

[3 p] **20.** Based on Routh–Hurwitz Criterion, this system is stable for $k$ selected as:

a) $k > -44.3$

b) $k > -0.011$

c) $-0.011 < k < 2.74$

d) $-44.3 < k < 10$

**20.** The system will be stable if all the elements in the first column have same sign. Thus the conditions for stability, in terms of $k$, are:

$$0.003 + 0.27k > 0 \Rightarrow k > -0.011$$
$$1.1 - 0.11k > 0 \Rightarrow k < 10$$
$$0.01k^2 - 1.55k + 4.17 > 0 \Rightarrow k < 2.74 \;\; \& \;\; k > 140.98$$
$$3.1 + 0.07k > 0 \Rightarrow k > -44.3$$

Combining above four constraints, the stable range of k can be found as:

$$-0.011 < k < 2.74$$

## Supplementary Material

Note: you *may* need some or none of these identities:

| No. | Property | Formula |
|---|---|---|
| 1 | Linearity | $\mathscr{Z}\{\alpha f_1(k) + \beta f_2(k)\} = \alpha F_1(z) + \beta F_2(z)$ |
| 2 | Time Delay | $\mathscr{Z}\{f(k-n)\} = z^{-n}F(z)$ |
| 3 | Time Advance | $\mathscr{Z}\{f(k+1)\} = zF(z) - zf(0)$<br>$\mathscr{Z}\{f(k+n)\} = z^nF(z) - z^nf(0) - z^{n-1}f(1)\cdots - zf(n-1)$ |
| 4 | Discrete-Time Convolution | $\mathscr{Z}\{f_1(k)^*f_2(k)\} = \mathscr{Z}\left\{\sum_{i=0}^{k} f_1(i)f_2(k-i)\right\} = F_1(z)F_2(z)$ |
| 5 | Multiplication by Exponential | $\mathscr{Z}\{a^{-k}f(k)\} = F(az)$ |
| 6 | Complex Differentiation | $\mathscr{Z}\{k^mf(k)\} = \left(-z\frac{d}{dz}\right)^m F(z)$ |
| 7 | Final Value Theorem | $f(\infty) = \mathscr{L}im_{k\to\infty} f(k) = \mathscr{L}im_{z\to 1}(1-z^{-1})F(z) = \mathscr{L}im_{z\to 1}(z-1)F(z)$ |
| 8 | Initial Value Theorem | $f(0) = \mathscr{L}im_{k\to 0} f(k) = \mathscr{L}im_{z\to\infty} F(z)$ |

$$e^{jx} = \cos x + j\sin x,$$
$$\cos x = \frac{e^{jx} + e^{-jx}}{2}, \quad \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$
$$x + j\,y = \sqrt{x^2 + y^2}\;\; e^{j\;\tan^{-1}(y/x)}$$
$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n\,s + \omega_n^2}, \quad \omega_d = \omega_n\sqrt{1-\xi^2}$$
$$T_r = \frac{\pi - \beta}{\omega_d}, \quad \beta = \tan^{-1}\frac{\omega_d}{\xi\omega_n}, \quad T_p = \frac{\pi}{\omega_d},$$
$$T_s = \left.\frac{3}{\xi\omega_n}\right|_{5\%} = \left.\frac{4}{\xi\omega_n}\right|_{2\%}, \qquad M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$$
$$k_0 = K_p\left(1 + \frac{T}{T_i} + \frac{T_d}{T}\right), k_1 = -K_p\left(1 + 2\frac{T_d}{T}\right),$$
$$k_2 = K_p\left(\frac{T_d}{T}\right)$$
$$\Phi = e^{Ah} = \mathscr{L}^{-1}\Big\{(sI-A)^{-1}\Big\}, \Gamma = \int_{t=0}^{h} e^{At}Bdt$$

اسم الطالب :

التاريخ : / /

رقم الجلوس

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نموذج الاختبار

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ختم الكنترول

| No. | Continuous Time | Laplace Transform | Discrete Time | $z$-Transform | |
|---|---|---|---|---|---|
| 1 | $\delta(t)$ | 1 | $\delta(k)$ | 1 | |
| 2 | $1(t)$ | $\frac{1}{s}$ | $1(k)$ | $\frac{z}{z-1}$ | |
| 3 | $t$ | $\frac{1}{s^2}$ | $kT$ | $\frac{zT}{(z-1)^2}$ | sampling $t$ gives $kT$, $z\{kT\} = T\,z\{k\}$ |
| 4 | $t^2$ | $\frac{2!}{s^3}$ | $(kT)^2$ | $\frac{z(z+1)T^2}{(z-1)^3}$ | |
| 5 | $t^3$ | $\frac{3!}{s^4}$ | $(kT)^3$ | $\frac{z(z^2+4z+1)T^3}{(z-1)^4}$ | |
| 6 | $e^{-\alpha t}$ | $\frac{1}{s+\alpha}$ | $a^k$ | $\frac{z}{z-a}$ | by setting $a = e^{-\alpha T}$. |
| 7 | $1-e^{-\alpha t}$ | $\frac{\alpha}{s(s+\alpha)}$ | $1-a^k$ | $\frac{(1-a)z}{(z-1)(z-a)}$ | |
| 8 | $e^{-\alpha t}-e^{-\beta t}$ | $\frac{\beta-\alpha}{(s+\alpha)(s+\beta)}$ | $a^k-b^k$ | $\frac{(a-b)z}{(z-a)(z-b)}$ | |
| 9 | $te^{-\alpha t}$ | $\frac{1}{(s+\alpha)^2}$ | $kTa^k$ | $\frac{az\,T}{(z-a)^2}$ | |
| 10 | $\sin(\omega_n t)$ | $\frac{\omega_n}{s^2+\omega_n^2}$ | $\sin(\omega_n kT)$ | $\frac{\sin(\omega_n T)z}{z^2-2\cos(\omega_n T)z+1}$ | |
| 11 | $\cos(\omega_n t)$ | $\frac{s}{s^2+\omega_n^2}$ | $\cos(\omega_n kT)$ | $\frac{z[z-\cos(\omega_n T)]}{z^2-2\cos(\omega_n T)z+1}$ | |
| 12 | $e^{-\zeta\omega_n t}\sin(\omega_d t)$ | $\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$ | $e^{-\zeta\omega_n kT}\sin(\omega_d kT)$ | $\frac{e^{-\zeta\omega_n T}\sin(\omega_d T)z}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$ | |
| 13 | $e^{-\zeta\omega_n t}\cos(\omega_d t)$ | $\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$ | $e^{-\zeta\omega_n kT}\cos(\omega_d kT)$ | $\frac{z[z-e^{-\zeta\omega_n T}\cos(\omega_d T)]}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$ | |
| 14 | $\sinh(\beta t)$ | $\frac{\beta}{s^2-\beta^2}$ | $\sinh(\beta kT)$ | $\frac{\sinh(\beta T)z}{z^2-2\cosh(\beta T)z+1}$ | |
| 15 | $\cosh(\beta t)$ | $\frac{s}{s^2-\beta^2}$ | $\cosh(\beta kT)$ | $\frac{z[z-\cosh(\beta T)]}{z^2-2\cosh(\beta T)z+1}$ | |