Digital Control

CSE421

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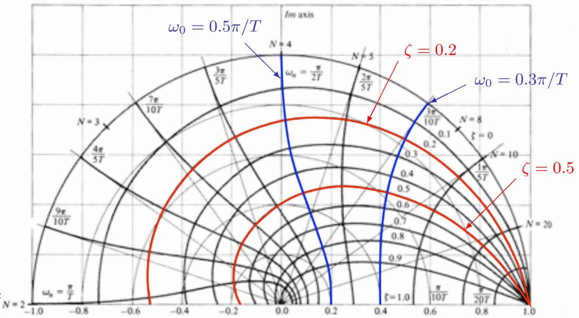
Lecture 14: **Pole placement (Regulator Problem)**



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Lecture 14

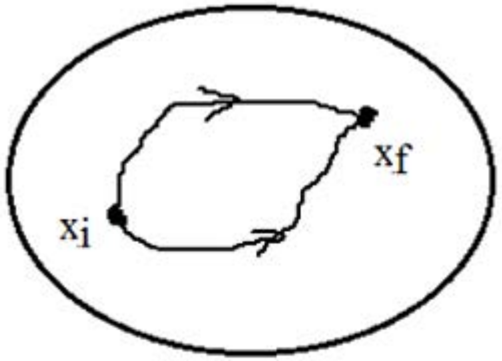
Pole placement

(Regulator Problem)

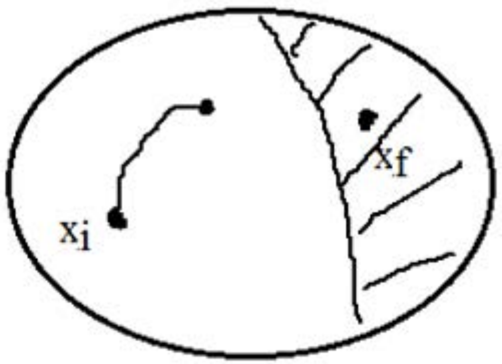
- Introduce the concept of controllability of dynamic systems.
- Explain pole placement using the state feedback approach.

Controllability

The system (Φ, Γ) is *controllable* if there is a control input sequence u(0), u(1) . . . u(n-1) that can move the system from an arbitrary initial state $\boldsymbol{x(0)} = \boldsymbol{x_i}$ to an arbitrary final state $\boldsymbol{x(n)} = \boldsymbol{x_f}$.



Controllable system



uncontrollable system: there is no input to move the system from state **xi** to **xf**

- The system state equation is

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

- Iterating this equation for ***n*** steps, we find

$$\begin{aligned}
x(1) &= \Phi x_i + \Gamma u(0) \\
x(2) &= \Phi x(1) + \Gamma u(1) \\
&= \Phi^2 x_i + \Phi\Gamma u(0) + \Gamma u(1) \\
&\vdots \\
x(n) &= \Phi^n x_i + \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma \end{bmatrix} \begin{bmatrix} u(n-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}
\end{aligned}$$

- To have ***x*(*n*)** = ***x**$_f$*, we must be able to solve the following ***n*** equations for the control sequence ***u*(*k*)**.

$$\begin{bmatrix}\Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma\end{bmatrix}\begin{bmatrix}u(n-1)\\ \vdots \\ u(1) \\ u(0)\end{bmatrix} = x_f - \Phi^n x_i$$

- Note that, for **single-input** systems, the following matrix is of size n x n.

$$\Delta_c = \begin{bmatrix}\Gamma & \Phi\Gamma & \cdots & \Phi^{n-1}\Gamma\end{bmatrix}$$

- The system is *controllable* if the above equations have a solution for the control sequence. This is achieved if the matrix $\boldsymbol{\Delta_c}$, called the *controllability* matrix, is nonsingular or, more generally, has full rank ***n.***

- Matrix rank = the number of linearly independent columns or rows.

Example 1

- Check the controllability of the system

$$x(k+1) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k)$$

The controllability matrix :

$$\Delta_c = [\Gamma \quad \Phi\Gamma] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow \text{rank} = 2 \Rightarrow \text{system is controllable}$$

- MATLAB functions to compute the controllability matrix and check its rank:

```
CO = ctrb(A,B)
rank(CO)
```

Example 2

- Check the controllability of the system

$$x(k+1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k)$$

$$\Delta_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \text{rank} = 1 \Rightarrow \text{system is not controllable}$$

- In order to understand why this system is not controllable, rewrite the state equation as:

$$x_1(k+1) = x_1(k) + u(k)$$

$$x_2(k+1) = x_2(k) + u(k)$$

- If $x_1(0) = x_2(0)$, then $x_1(k) = x_2(k)$ for all k. Suppose that we want $x_{1f} \neq x_{2f}$, there is no input sequence to achieve this final state. Hence, this system is not completely state controllable.

State feedback & Pole Placement

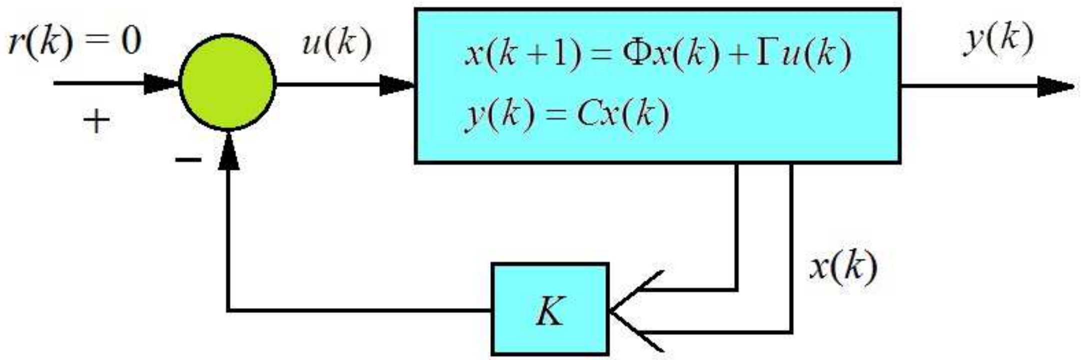
- In the state feedback approach, instead of using controllers with fixed configuration in the forward path,
    - control signal ***u*(*k*)** is a calculated as a *linear combination of the measured state variables*.

- Using state feedback, the poles or eigenvalues of the closed-loop system can be placed at specified locations (also called pole assignment or allocation).

- Poles can arbitrarily be placed if and only if the system is *controllable*.

State feedback block diagram

- A linear system (Φ,Γ,C) with constant state feedback gain matrix ***K*** :



- The control signal ***u*(*k*)** is simply a linear combination of all state variables:

$$u(k) = -Kx(k).$$

- It is assumed, for now, that the reference input ***r*(*k*)** is zero (i.e. a **regulator** problem) and that all the states are available for feedback – that is, we have access to the complete state **x**(*k*) for all *k*.

- The objective of the state feedback design is to determine the matrix ***K***
  - ➢ for single-input systems, ***K*** is a row vector of size **1 x *n***.

- The linear system and the feedback control law are given by:

$$x(k+1) = \Phi x(k) + \Gamma u(k),$$
$$y(k) = Cx(k)$$
$$u(k) = -Kx(k)$$

- Substituting by ***u*(*k*)** in the state equation yields the closed-loop state equation

$$x(k+1) = \overbrace{(\Phi - \Gamma K)}^{\Phi_{cl}} x(k)$$
$$y(k) = Cx(k)$$

Where $\Phi_{cl} = \Phi - \Gamma K$, is the state matrix of the closed-loop system.

Theorem: State feedback

If the pair (Φ,Γ) is **controllable**, then

there exists a feedback gain matrix *K* that arbitrarily assigns the closed-loop system poles to any set $[\lambda_1, \lambda_2, \ldots, \lambda_n]$.

That is the eigenvalues of the closed-loop state matrix

$$\Phi_{cl} = \Phi - \Gamma K$$

can be arbitrarily assigned.

pole placement by equating coefficients

1. Evaluate the desired characteristic polynomial from the specified eigenvalues using the expression

$$\prod_{i=1}^{n}(\lambda-\lambda_i)=0$$

2. Evaluate the closed-loop characteristic polynomial using the expression

$$\det[\lambda I-(\Phi-\Gamma K)]=0.$$

3. Compare the two polynomials in 1 & 2 to get the entries of the gain matrix ***K***.

Example 3

Assign the eigenvalues {0.3± j0.2} to the pair

$$\Phi = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution:

- For the given eigenvalues, the desired characteristic polynomial is

$$\Delta = (\lambda - 0.3 + j0.2)(\lambda - 0.3 - j0.2) = \lambda^2 - 0.6\lambda + 0.13$$

- The closed-loop state matrix is

$$\Phi_{cl} = \Phi - \Gamma K = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 - k_1 & 4 - k_2 \end{pmatrix}$$

- The closed-loop characteristic polynomial is

$$|\lambda I - A_{cl}| = \begin{vmatrix} \lambda & -1 \\ -(3-k_1) & \lambda-(4-k_2) \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (4-k_2)\lambda - (3-k_1) = 0$$

- Comparing with the desired characteristic polynomial,

$$\Delta = \lambda^2 - 0.6\lambda + 0.13$$

- Gives

$$k_1 = 3.13, \quad k_2 = 3.4$$

- To check, use MATLAB commands:

```
>> A = [0, 1; 3, 4];
>> B = [0;  1];
>> poles = [0.3+j*0.2, 0.3-j*0.2];
>> K = place(A, B, poles)
```

$K = 3.13 \quad 3.40$

Controllable canonical form

- The algebra for finding the state feedback gain matrix ***K*** for systems with n > 2 becomes quite tedious.
- However, it is specially simple if the system matrices happen to be in the **controllable canonical form**.

- For a third order system, this form is:

$$\Phi = \begin{bmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

Example 4

Design a feedback controller for the pair

$$\Phi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

to obtain the eigenvalues {0.1, 0.4±j0.4}.

Answer

- For the given eigenvalues, the desired characteristic polynomial is

$$\Delta = (\lambda - 0.1)(\lambda - 0.4 + j0.4)(\lambda - 0.4 - j0.4)$$
$$= \lambda^3 - 0.9\lambda^2 + 0.4\lambda - 0.032.$$

- The closed-loop state matrix is

$$\Phi_{cl} = \Phi - \Gamma K = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix} = \begin{pmatrix} 1-k_1 & 2-k_2 & 3-k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left|\lambda I - \Phi_{cl}\right| = \begin{vmatrix} \lambda - 1 + k_1 & k_2 - 2 & k_3 - 3 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda^2(\lambda - 1 + k_1) + \lambda(k_2 - 2) + (k_3 - 3)$$

$$= \lambda^3 + (k_1 - 1)\lambda^2 + (k_2 - 2)\lambda + (k_3 - 3)$$

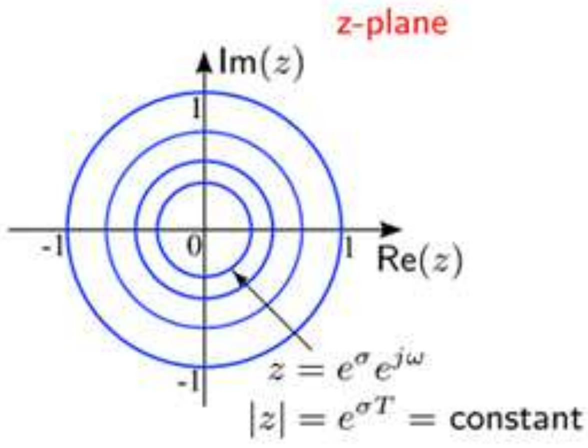
$$\Rightarrow k_1 = 0.1, \quad k_2 = 2.4, \quad k_3 = 2.968$$

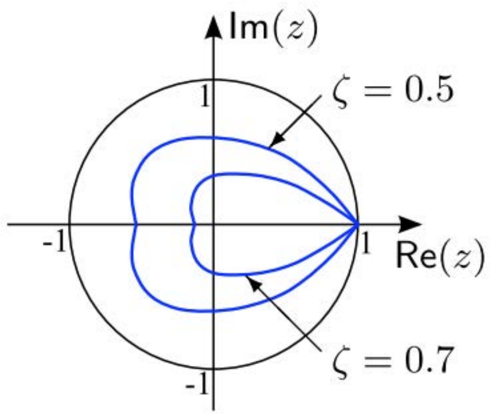
Selecting desired pole locations

- The locations of the poles (eigenvalues) in the z-plane are directly related to the transient response of the system.

- For example, the smaller is the distance of the pole **from the origin**, the **faster** is the response associated with it.

- Also, the contours of constant damping ratio (which determines percent overshoot) are spirals as shown.





Note on selecting desired pole locations!

- Seeking a closed-loop response that is much faster than the slowest component response (often the plant) will lead to high gains for the state feedback matrix and consequently to a high control effort which may saturate one or more components.

- For example, if all closed-loop eigenvalues are placed at the origin of the complex plane (i.e. deadbeat control), the resulting control law can assume unacceptably high values.

Thanks for your attention.

Questions?

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Robotics Research Interest Group (zuR^2IG)
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