Digital Control

CSE421

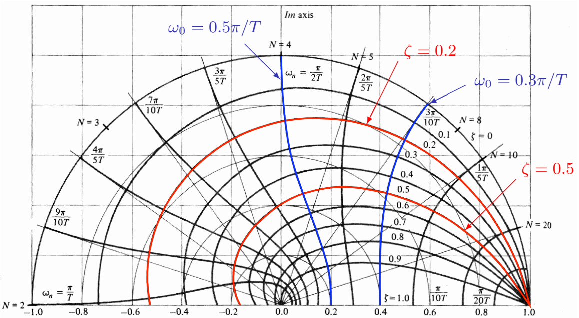
Asst. Prof. Dr.Ing.

Mohammed Nour A. Ahmed

mnahmed@eng.zu.edu.eg

https://mnourgwad.github.io

Lecture 12: **State-Space Models**



Zagazig University | Faculty of Engineering | Computer and Systems Engineering Department | Zagazig, Egypt

Lecture: 12

State-Space Models

- Continuous-Time State-Space Models Review

Continuous-Time State-Space Models Review

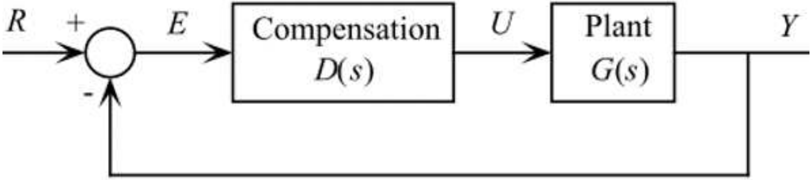
- The idea of state space comes from the **state–variable** method of describing differential equations.
- In this method, dynamic systems are described by a set of first-order differential equations in variables called the **states**,
  - the **solution** may be thought of as a **trajectory in the state space**.

Modern vs. classical control

- In control systems, the use of **state-space** approach is often referred to as **modern control**, while the use of **transform** methods is termed **classical** control.
- However, the state-space formulation of differential equations is over 100 years old, being introduced to control design in the late 1950s, so it may be misleading to consider it "*modern*."
- It is better to refer to the two approaches as **transform method** and **state-space method**.

Goal of state space description

- Although the state space approach is so different from transform-based approach, they both have the same goal which is to find a controller, $D(s)$, that yields acceptable performance.



State Space Equation

- The general form of a state space description is:

$$\dot{x} = f(x, u, t) \quad \text{state equation}$$
$$y = g(x, u, t) \quad \text{output equation}$$

- Where
  - $x_{n\times 1}$ is the state vector,
  - $u_{m\times 1}$ is the input vector,
  - $y_{p\times 1}$ is the output
- The functions f and g are possibly nonlinear.

State equation and output equation

- For **linear time-invariant** systems, the state space description can be written in the following matrix form:

$$\dot{x} = A\,x + B\,u \quad \text{state equation}$$
$$y = C\,x + D\,u \quad \text{output equation}$$

  - $A_{n\times n}$: system matrix,
  - $B_{n\times m}$: input matrix,
  - $C_{p\times n}$: output matrix,
  - $D_{p\times m}$: feed-forward matrix
    D connects the input directly to the output. Usually, it is zero for physical systems.

State space description

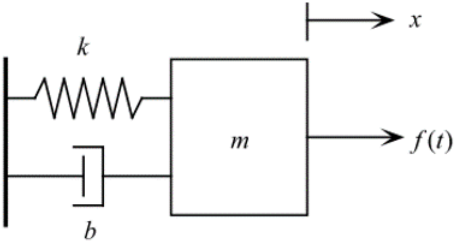
- Any lumped-parameter dynamic system can be described using a set of first-order ordinary differential equations, where the dependent variables are the state variables.
- knowledge of state variables at some initial time, plus knowledge of input from that time on, is sufficient to predict the system behavior.
- Variables with which initial conditions are necessary are by definition state variables.
- **State variables** also define the **energy stored** within the system.
- Valid state variables: position and velocity (mechanical), voltage and current (electrical), and system **output** and its **derivatives**.
- Any linear combination of state variables is itself a valid state variable. There are infinite number of choices.
- The set of state variables is collectively called the **state vector** or simply the state.

State equation and output equation

Example 1

Example

Write the the state-space equations for the following spring-mass-damper system. The input is the applied force $f(t)$ and the output is the position $x(t)$.



- The equation of motion for this system is:

$$f(t) - b\dot{x} - kx = m\ddot{x} \quad \Rightarrow \quad m\ddot{x} + b\dot{x} + kx = f(t)$$

- Position and velocity are valid state variables, hence

$$x_1 = x \quad \Rightarrow \quad \dot{x}_1 = x_2 \qquad x_2 = \dot{x} \quad \Rightarrow \quad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}f(t)$$

- Therefore, the matrices of the state-space description are:

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

- Some MATLAB functions to obtain the dynamic response of state-space systems are:

```
SYS = ss(A,B,C,D)
[Y,T,X] = step(SYS)
[Y,T,X] = initial(SYS,X0)
[Y,X] = lsim(A,B,C,D,U,T,X0)
```

Transfer function from state-space description

- State space and transfer function are two different, but equivalent, representations of the system. It is possible to convert from one form to the other.
- Given the state space description of a system, its transfer function is obtained as:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

- MATLAB commands to convert between state space and transfer function:

```
[num,den] = ss2tf(A,B,C,D)
[A,B,C,D] = tf2ss(num,den)
```

Transfer function from state-space description

Example 2

Example

A state space description of a spring-mass-damper system is given by the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Find the system transfer function.

- The transfer function is given by:

$$G(s) = C(sI - A)^{-1}B$$

$$sI - A = \begin{bmatrix} s & -1 \\ 4 & s+2 \end{bmatrix} \quad \Rightarrow \quad (sI - A)^{-1} = \frac{1}{s^2 + 2s + 4} \begin{bmatrix} s+2 & 1 \\ -4 & s \end{bmatrix}$$

$$G(s) = \frac{1}{s^2 + 2s + 4} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ -4 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 + 2s + 4}$$

Advantages of state space description

- Well-suited to computer calculation.
- Controller design approach for multiple-inputs multiple-outputs (MIMO) systems is the same as for single-input single-output (SISO) systems.
- With the state-space formulation, the internal behavior of the system is exposed, rather than the input-output modeling of transform (i.e. transfer function based) methods.

Thanks for your attention.

Questions?

Asst. Prof. Dr.Ing.

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Robotics Research Interest Group (zuR^2IG)
Zagazig University | Faculty of Engineering | Computer and Systems Engineering Department | Zagazig, Egypt