Digital Control

CSE421

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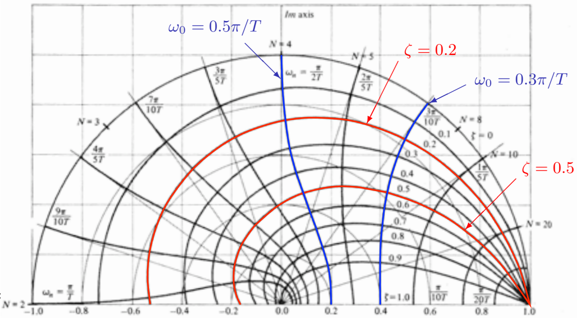
Lecture 9: **Discrete Controller Design (Pole Placement)**



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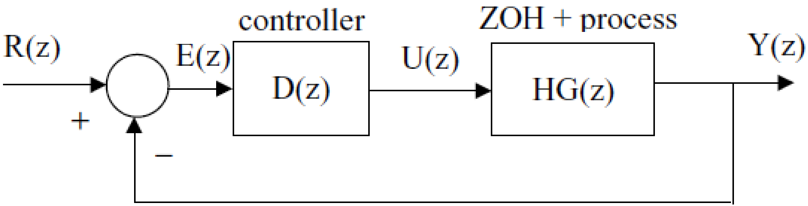
Lecture: 9

Discrete Controller Design (Pole Placement)

- procedure for discrete controller design
- Review step response characteristics and relate them to system pole locations
- Introduce pole placement method for controller design.

Procedure for discrete-time controller design



1. Derive the transfer function of the plant (process).
2. Transform the process transfer function into z-plane.
3. Design a suitable digital controller in z-plane.
4. Implement the controller algorithm on a digital computer.

Digital controllers

- The closed-loop transfer function of the system is:

$$\frac{Y(z)}{R(z)} = \frac{D(z)GH(z)}{1 + D(z)GH(z)}$$

- Suppose it is desired that the closed-loop transfer function be:

$$T(z) = \frac{Y(z)}{R(z)}$$

- Then, the controller required to achieve T(z) is given by:

$$D(z) = \frac{1}{GH(z)} \frac{T(z)}{1 - T(z)}$$

- Note that the **controller** $D(z)$ **must be realizable**, that is, the degree of the numerator must not exceed that of the denominator.

Time domain specifications

- Most commonly used time domain performance measures refer to a 2^{nd} order system:

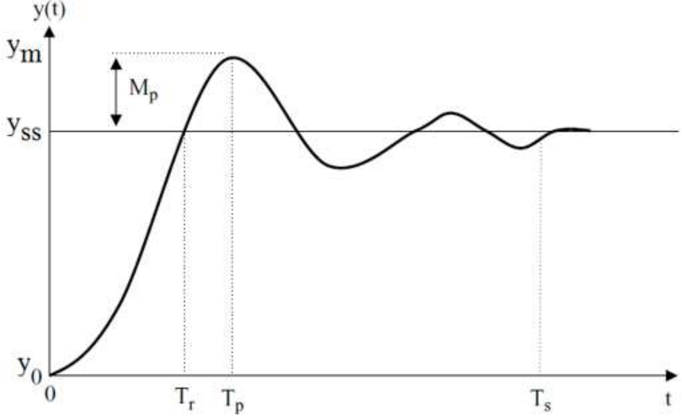
$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n\, s + \omega_n^2}$$

ω_n	undamped natural frequency
ξ	damping ratio

- The performance of a control system is usually characterized by its step response.
  - step input is easy to generate and gives the system a nonzero steady-state condition, which can be measured.

Time domain specifications

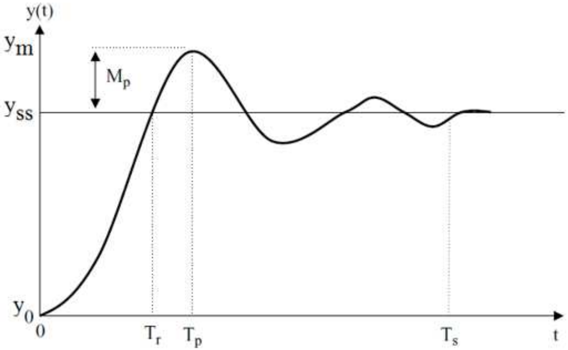
- The typical response of a 2^{nd} order system when excited with a unit step input



- In this response, the performance indices are usually defined: **rise time; peak time; settling time; maximum overshoot; and steady-state error**.

Time domain specifications

Rise time (T_r)

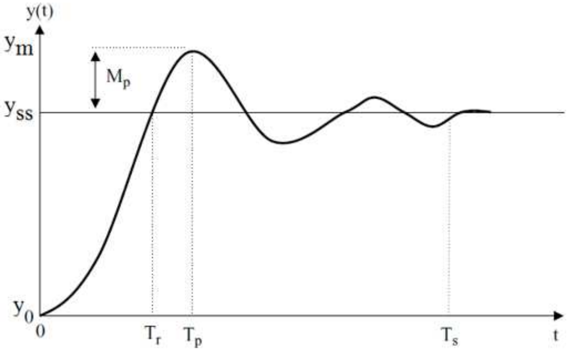


- time required for the response to go from 0 to 100% of its final value.
- a measure of speed of response (smaller rise time ⇒ faster response)

$$T_r = \frac{\pi - \beta}{\omega_d}, \quad \beta = \tan^{-1}\frac{\omega_d}{\xi\omega_n}, \quad \omega_d = \omega_n\sqrt{1-\xi^2}$$

Time domain specifications

Peak time (T_p)

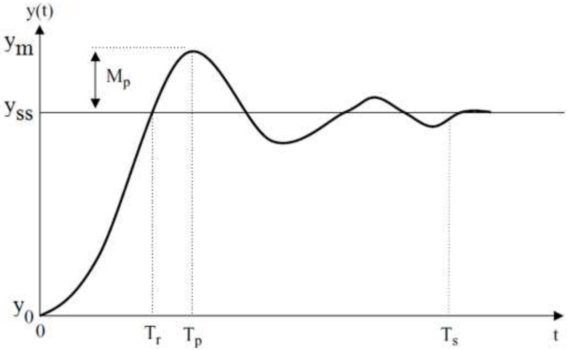


- time required for the response to reach its peak value.
- response is faster when peak time is smaller, but with higher overshoot.

$$T_p = \frac{\pi}{\omega_d}$$

Time domain specifications

Settling time (T_s)

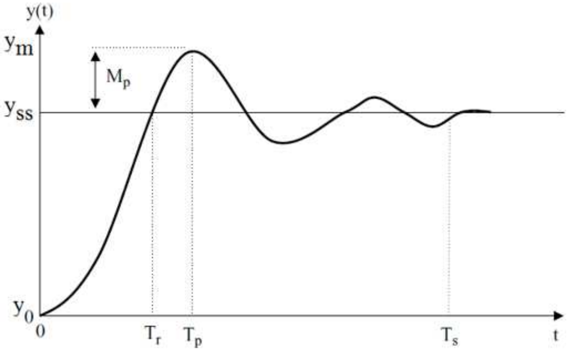


- time required for response to reach and stay within a range (2 or 5%) about final value.

$$T_s = \frac{3}{\xi\omega_n} \quad \text{(for 5\%)} \quad T_s = \frac{4}{\xi\omega_n} \quad \text{(for 2\%)}$$

Time domain specifications

Percent Overshoot (M_p)



- The percent overshoot is defined as:

$$M_p = \frac{y_m - y_{ss}}{y_{ss} - y_0} \times 100\%,$$

y_0 initial, y_m maximum, and y_{ss} is steady state (final) values of the step response,

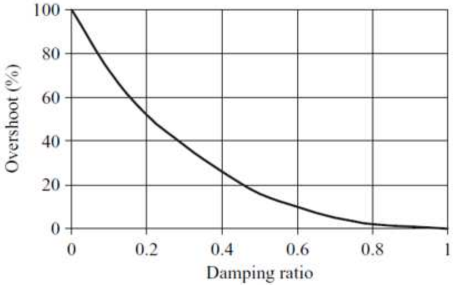
Time domain specifications

Percent Overshoot (M_p)

- For 2^{nd} order system, the percent overshoot is calculated as:

$$M_p = \exp(-\xi\pi/\sqrt{1-\xi^2})$$

- The amount of overshoot depends on the damping ratio (ξ) and directly indicates the relative stability of the system.
  - The lower is the damping ratio, the higher the is maximum overshoot.



The steady-state error (e_{ss})

- difference between the reference input and the output response at steady-state.
- Small e_{ss} is required in most control systems. However, in some systems such as position control, it is important to have zero e_{ss}.
- e_{ss} can be found using the final value theorem:
  If the Laplace transform of the output is $Y(s)$, then final (steady-state) value is given by:

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

- Hence, for a unit step input, e_{ss} is given as:

$$e_{ss} = 1 - \lim_{s \to 0} sY(s)$$

Time domain specifications

Example

Determine the step response **performance indices** of the system with the following closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{1}{s^2 + s + 1}$$

- Comparing the given system with the standard 2^{nd} order transfer function:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- We find that $\omega_n = 1$ rad/s and $\xi = 0.5$. Thus, the damped natural frequency is:

$$\omega_d = \omega_n\sqrt{1 - \xi^2} = 0.866$$

Time domain specifications

Solution of Example

- percent overshoot is:

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.16 = 16\%$$

- peak time is:

$$T_p = \frac{\pi}{\omega_d} = 3.627$$

$$\beta = \tan^{-1}\frac{\omega_d}{\xi\omega_n} = 1.047,$$

$$T_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 1.047}{0.866} = 2.42 \text{ sec},$$

$$T_s = \frac{4}{\xi\omega_n} = 8 \text{ sec},$$

$$e_{ss} = 1 - \lim_{s\to 0} sY(s) = 1 - \lim_{s\to 0} s\frac{1}{s(s^2+s+1)} = 0$$

Pole placement approach

- As noticed, the step response performance indices are functions of (ξ, ω_n). Also (ξ, ω_n) determine system poles:

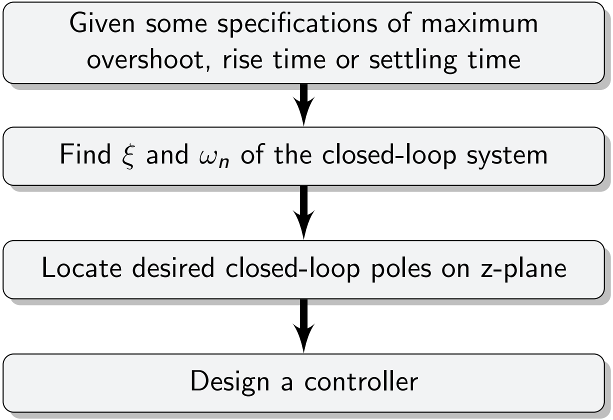
$$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

- Therefore, the **response** of a system is determined by the position of its **poles**.

placing the closed-loop poles at **good** locations, we can shape the response of the system and achieve desired time response characteristics.

- Although the previous analysis is conducted for 2^{nd} order continuous-time system, the pole placement approach is also applicable for discrete-time systems.
  - thanks to the mapping between s- and z-plane poles $(z = e^{sT})$ provided that the sampling interval is chosen sufficiently small.

Pole placement design procedure



Example

The open-loop transfer function of a system together with a ZOH is given by:

$$HG(z) = \frac{0.03(z + 0.75)}{z^2 - 1.5z + 0.5}, \quad T = 0.2$$

Design a digital controller so that for the closed-loop system:

- percent overshoot is 9.5%,
- settling time (2%) is 1.777 sec,
- steady-state error to a step input is zero and to a ramp input is 0.2.

- From the required percent overshoot and settling time:

$$M_p = \exp(-\xi\pi/\sqrt{1-\xi^2}) = 0.095 \quad \Rightarrow \quad \xi = 0.6$$
$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{0.6\ \omega_n} = 1.777 \quad \Rightarrow \quad \omega_n = 3.75 \text{ rad/sec}$$

- The damped natural frequency:

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 3 \text{ rad/sec}$$

- Hence, the required pole positions in the z-plane are:

$$\begin{aligned} z_{1,2} &= e^{Ts} = e^{-\xi\omega_n T \pm j\omega_d T} \\ &= e^{-\zeta\omega_n T}\left[\cos(\omega_d T) \pm j\sin(\omega_d T)\right] \\ &= e^{-0.6\times3.75\times0.2}[\cos(0.2\times3) \pm j\sin(0.2\times3)] = 0.526 \pm j0.36. \end{aligned}$$

Pole placement design procedure

Solution of Example

- The desired closed-loop transfer function is thus given by:

$$
\begin{aligned}
T(z) &= \frac{N(z)}{(z - 0.526 + j0.36)(z - 0.526 - j0.36)} \\
&= \frac{N(z)}{z^2 - 1.05z + 0.405}.
\end{aligned}
$$

- Before we proceed, we must determine the degree of $N(z)$. This degree must be ≤ 2 so that $T(z)$ is realizable.
- If we choose $N(z)$ to be exactly of order 2, this means that the closed-loop will start its response instantaneously without delay.

Pole placement design procedure

Solution of Example

- However, by recalling the transfer function of the process:

$$HG(z) = \frac{0.03(z + 0.75)}{z^2 - 1.5z + 0.5}$$

- We can realize that the process itself has a delay of 1 sample (the order of the denominator exceeds that of the numerator by 1), so the closed-loop must have, at least, the same delay.
- With this reasoning, the desired transfer function T(z) is chosen as:

$$T(z) = \frac{b_1 z + b_0}{z^2 - 1.05z + 0.405}$$

- Note that this choice insures the realizability of the controller we are designing.

Pole placement design procedure

Solution of Example

The parameters b_1 and b_0 can be determined from the steady-state requirements.

- For input $R(z)$, the error is given by:

$$\begin{aligned} E(z) &= R(z) - Y(z) \\ &= R(z) - T(z)R(z) \\ &= R(z)[1 - T(z)] \end{aligned}$$

- So, the steady-state error is given by:

$$e_{ss} = \lim_{z \to 1} (1 - z^{-1}) R(z) [1 - T(z)]$$

- From given specifications: for a unit-step input, $e_{ss} = 0$, so

$$\begin{aligned} e_{ss} &= \lim_{z \to 1} (1 - z^{-1}) \frac{z}{z-1} [1 - T(z)] = 0 \\ &= 1 - T(1) = 0 \\ &= 1 - \left. \frac{b_1 z + b_0}{z^2 - 1.05z + 0.405} \right|_{z=1} = 0 \\ &= 1 - \frac{b_1 + b_0}{0.355} = 0 \end{aligned}$$

$$\Rightarrow b_1 + b_0 = 0.355$$

Pole placement design procedure

Solution of Example

- Also, from the given specifications, it is required that, for a unit-ramp input, $e_{ss} = 0.2$:

$$e_{ss} = \lim_{z \to 1} (1 - z^{-1}) \frac{Tz}{(z-1)^2} [1 - T(z)]$$

$$= \lim_{z \to 1} \frac{0.2}{z-1} \left[1 - \frac{b_1 z + b_0}{z^2 - 1.05z + 0.405} \right] = \frac{0}{0}$$

- Hence, we need to apply **L'Hospital rule** which gives:

$$e_{ss} = 0.2 \frac{0.95 - b_1}{0.355} = 0.2$$

- From which we can find that:

$$b_1 = 0.595 \quad b_0 = -0.240$$

Pole placement design procedure

Solution of Example

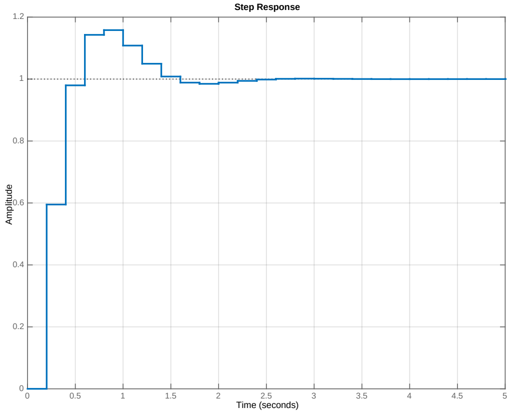
- Hence, the desired closed-loop transfer function is:

$$T(z) = \frac{0.595z - 0.24}{z^2 - 1.05z + 0.405}$$

- The controller that achieves this closed-loop transfer function is given by:

$$\begin{aligned} D(z) &= \frac{1}{HG(z)} \frac{T(z)}{1 - T(z)} \\ &= \frac{z^2 - 1.5z + 0.5}{0.03(z + 0.75)} \frac{0.595z - 0.24}{z^2 - 1.645z + 0.645}, \\ &= \frac{0.595z^3 - 1.133z^2 + 0.657z - 0.120}{0.03z^3 - 0.027z^2 - 0.018z + 0.015}. \end{aligned}$$

closed-loop system step response with the designed controller:



Thanks for your attention.

Questions?

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