Digital Control

CSE421

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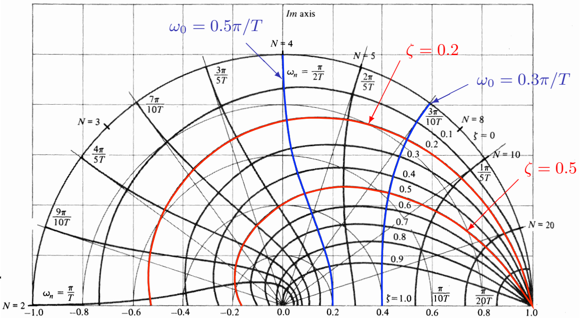
Lecture 8: **Stability of Discrete Systems**



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Lecture: 8

Stability of Discrete Systems

- Factorization
- Jury Test
- Routh–Hurwitz Criterion

Stability of Discrete Systems

- Suppose that we have the following transfer function of a closed-loop discrete-time system:

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} = \frac{N(z)}{D(z)}$$

- The system is **stable** if **all** poles* lie inside the unit circle in z-plane.

*roots of the characteristic equation $D(z) = 0$

Stability of Discrete Systems

There are several methods to check the stability of a discrete-time system such as:

- Factorizing $D(z)$ and finding its roots.
- Jury Test.
- Routh–Hurwitz criterion .

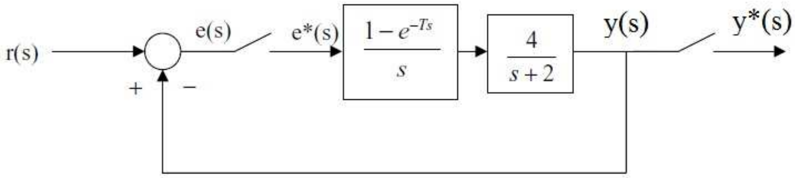
Factorizing the Characteristic Equation

- The direct method to check system stability is to factorize the characteristic equation,
  - determine its roots, and check if their **magnitudes** are all less than 1.
- it is **not usually easy** to factorize the characteristic equation by hand
- we can use MATLAB command `roots`.

Example 1

Example

Check the stability of the following closed-loop discrete system. Assume that $T = 1$ s.



- The transfer function of the closed-loop system is:

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$

Example 1

- Where

$$G(z) = \mathscr{Z}\left\{\frac{1-e^{-Ts}}{s}\frac{4}{s+2}\right\}$$

$$= (1-z^{-1})\frac{2z(1-e^{-2T})}{(z-1)(z-e^{-2T})} = \left.\frac{2(1-e^{-2T})}{(z-e^{-2T})}\right|_{T=1\text{ sec}} = \frac{1.729}{z-0.135}$$

- The characteristic equation is thus:

$$1 + G(z) = 0$$

$$z + 1.594 = 0$$

$$z = -1.594$$

$$|z| > 1 \quad \Rightarrow$$ system is **unstable**

Example 2

Example

In the previous example, find the value of T for which the system is stable.

- From the previous example, we found:

$$G(z) = \frac{2(1 - e^{-2T})}{(z - e^{-2T})}$$

- The characteristic equation is:

$$\begin{aligned} 1 + G(z) &= 0 \\ z - 3e^{-2T} + 2 &= 0 \\ z &= 3e^{-2T} - 2 \end{aligned}$$

Example 2

- For stability, the condition $|z| < 1$ must be satisfied;

$$|z| = |3e^{-2T} - 2| < 1$$

$$-1 < 3e^{-2T} - 2 < 1$$

$$\ln\left(\frac{1}{3}\right) < -2T < 0$$

$$-0.5\ln\left(\frac{1}{3}\right) > T > 0$$

$$0 < T < 0.549$$

- Thus the system is stable as long as $T < 0.549$.

Jury Stability Test

- Jury stability test[†] is similar to Routh–Hurwitz stability criterion used for continuous systems.
- in Jury test, the characteristic equation of a discrete system of order n is expressed as:

$$F(z) = a_n\, z^n + a_{n-1}\, z^{n-1} + \cdots + a_2\, z^2 + a_1\, z + a_0 = 0, \quad \textit{where} \quad a_n > 0$$

- Then, form the table:

	z^0	z^1	z^2	$\cdots$	z^{n-1}	z^n
1	a_0	a_1	a_2	$\cdots$	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_1	a_0
3	b_0	b_1	b_2	$\cdots$	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_0	
5	c_0	c_1	c_2	$\cdots$		
6	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$		
.	.	.	.	$\cdots$		
$2n-3$	r_0	r_1	r_2			

[†] it is called Jury test for real coefficients and **Schur-Cohn** test for complex coefficients

Jury Stability Test

	z^0	z^1	z^2	$\cdots$	z^{n-1}	z^n
1	a_0	a_1	a_2	$\cdots$	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_1	a_0
3	b_0	b_1	b_2	$\cdots$	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_0	
5	c_0	c_1	c_2	$\cdots$		
6	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$		
.	.	.	.	$\cdots$		
$2n-3$	r_0	r_1	r_2			

The elements of this array are defined as follows:

- elements of **even**-numbered row are the elements of the preceding row, in reverse order.
- elements of the **odd**–numbered rows are defined as given by $b_k, c_k, \cdots$

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}, \quad \cdots$$

Jury Stability Test

Another way to calculate odd row elements

- The 3^{rd} row is calculated by subtracting ($\frac{a_n}{a_0} \times 2^{nd}$ row elem.) from the 1^{st} row elem.
  - then for 5^{th} and after, the coefficient changes (i.e. $\frac{b_{n-1}}{b_0}$).

	z^0	z^1	z^2	$\cdots$	z^{n-1}	z^n
1	a_0	a_1	a_2	$\cdots$	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_1	a_0
3	$\left(a_0 - a_n \frac{a_n}{a_0}\right)$	$\left(a_1 - a_{n-1} \frac{a_n}{a_0}\right)$	$\left(a_2 - a_{n-2} \frac{a_n}{a_0}\right)$	$\cdots$	$\left(a_{n-1} - a_1 \frac{a_n}{a_0}\right)$	0
4	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_0	
5	c_0	c_1	c_2	$\cdots$		
6	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$		
.	.	.	.	$\cdots$		
$2n-3$	r_0	r_1	r_2			

The expansion of the table is continued in this manner until a row containing only one non zero element is reached.

Jury Stability Test

The *necessary and sufficient* conditions for the characteristic equation to have all roots inside the unit circle are given as:

(I)

$$F(1) > 0,$$
$$(-1)^n F(-1) > 0,$$
$$|a_0| < a_n,$$

(II)

$$|b_0| > |b_{n-1}|$$
$$|c_0| > |c_{n-2}|$$
$$|d_0| > |d_{n-3}|$$
$$\vdots$$

Jury Test is applied as follows:

- Check the three conditions (I) and stop if any of them is not satisfied.
- Construct Jury array and check the conditions (II) . Stop if any condition is not satisfied.

Jury Test

2^{nd} order polynomial

- For 2^{nd} order characteristic equation:

$$F(z) = a_2 z^2 + a_1 z + a_0 = 0, \quad where \quad a_2 > 0$$

- Jury Test reduces to the following simple rules: no roots of the system characteristic equation will be on or outside the unit circle provided that:

$$F(1) > 0, \quad F(-1) > 0, \quad |a_0| < a_2$$

- Conditions (II) **reduce** to these conditions for first and second-order systems, respectively as the Jury table is simply one row.

Jury Test

3^{rd} order polynomial

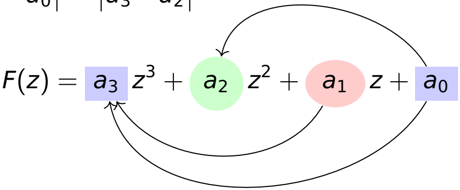
- For 3^{rd} order characteristic equation:

$$F(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0, \quad where \quad a_3 > 0$$

- Jury Test reduces to the following simple rules:

$$F(1) > 0, \quad F(-1) < 0, \quad |a_0| < a_3,$$

$$\begin{vmatrix} a_0 & a_3 \\ a_3 & a_0 \end{vmatrix} > \begin{vmatrix} a_0 & a_1 \\ a_3 & a_2 \end{vmatrix} \Rightarrow (a_0^2 - a_3^2) > (a_0\, a_2 - a_1\, a_3)$$



Detailed Example

Example

Test the stability of the polynomial:

$$F(z) = z^5 + 2.6\, z^4 - 0.56\, z^3 - 2.05\, z^2 + 0.0775\, z + 0.35 = 0$$

- We compute the entries of the Jury table using the coefficients of the polynomial

	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	−2.05	−0.56	2.6	1
2	1	2.6	−0.56	−2.05	0.0775	0.35
3	−0.8775	−2.5729	−0.1575	1.854	0.8325	
4	0.8325	1.854	−0.1575	−2.5729	−0.8775	
5	0.0770	0.7143	0.2693	0.5151		
6	0.5151	0.2693	0.7143	0.0770		
7	−0.2593	−0.0837	−0.3472			

Detailed Example

- The first two conditions require the evaluation of $F(z)$ at $z = \pm 1$:
  1. $F(1) = 1 + 2.6 - 0.56 - 2.05 + 0.0775 - 0.35 = 1.4175 > 0$ ✓
  2. $(-1)^5 F(-1) = (-1)(-1 + 2.6 + 0.56 - 2.05 - 0.0775 + 0.35) = -0.3825 < 0$ ✗
- Conditions 3 through 6 can be checked quickly using the entries of Jury table 1^{st} column:
  1. $|0.35| < 1$ ✓
  2. $|-0.8775| > |0.8325|$ ✓
  3. $|0.0770| < |0.5151|$ ✗
  4. $|-0.2593| < |-0.3472|$ ✗
- Conditions 2, 5, and 6 are violated $\Rightarrow$ there are roots on or outside the unit circle.
- violation of condition 2 is sufficient to conclude the instability of $F(z)$.
- In fact, the polynomial can be factored as

$$F(z) = (z - 0.7)(z - 0.5)(z + 0.5)(z + 0.8)(z + 2.5) = 0$$

  and has a root at -2.5 outside the unit circle.
- **Note**: that the number of conditions violated is **not equal** to the number of roots outside the unit circle

Observations on Jury Table

Based on the Jury table and the Jury stability conditions, we make the following observations:

1. 1^{st} row of the Jury table is a listing of $F(z)$ coefficients in order of **increasing** power of z.
2. The table has $2n - 3$ rows (always odd)
3. The last row always has 3 elements.
   - Once we get to a row with 2 members, we can stop constructing the array.
4. This test doesn't have sense if N=1, but in this case you know the pole!
5. coefficients of each **even** row are the same as the odd row directly above it with its order **reversed**.
6. There are $n + 1$ conditions in (II) that correspond to the $n + 1$ coefficients of $F(z)$.
7. Conditions 3 through $n + 1$ of (II) are calculated using the coefficient of 1^{st} column together with the last coefficient of the preceding row.
   - The middle coefficient of the last row is never used and need not be calculated.

Observations on Jury Table

8. Conditions 1 and 2 of (II) are **directly** calculated from $F(z)$.
   - If one of the 1^{st} two conditions is violated, we conclude that $F(z)$ has roots on or outside the unit circle **without** the need to construct the Jury table or test the remaining conditions.
9. Condition 3 of (II), with $a_n = 1$, requires the constant term of the polynomial to be less than unity in magnitude.
   - the constant term is the product of all roots
   - it must be smaller than unity for all roots to be inside the unit circle.
10. For higher-order systems, applying the Jury test by hand is laborious,
    - to test its stability, it is preferable to use computer software.
11. If the coefficients of the polynomial are functions of system parameters, the Jury test can be used to obtain their stability ranges.

Example 3

Example

The closed-loop transfer function of a system is given by

$$\frac{G(z)}{1+G(z)}, \quad \text{where } G(z) = \frac{0.2z+0.5}{z^2-1.2z+0.2}$$

Determine the stability of this system using Jury Test.

- The characteristic equation is:

$$1+G(z) = 0$$
$$1+\frac{0.2z+0.5}{z^2-1.2z+0.2} = 0$$
$$z^2 - z + 0.7 = 0$$

- Applying Jury Test:

$$F(1) = 0.7 > 0, \quad F(-1) = 2.7 > 0,$$
$$|a_0| = 0.7 < 1 = a_2$$

- All conditions are satisfied, so the system is **stable**.

Example 4

Example

Determine the stability of a system having the following characteristic equation:

$$F(z) = z^3 - 2z^2 + 1.4z - 0.1 = 0$$

- Applying Jury test:

$$a_3 = 1, a_2 = -2, a_1 = 1.4, a_0 = -0.1$$
$$F(1) = 0.3 > 0, \quad F(-1) = -4.5 < 0, \quad |a_0| = 0.1 < 1 = a_3$$

- The first conditions are satisfied. Applying the other condition:

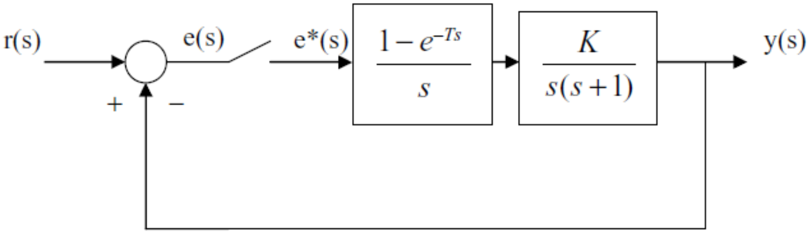
$$\begin{vmatrix} -0.1 & 1 \\ 1 & -0.1 \end{vmatrix} = -0.99 \quad \text{and} \quad \begin{vmatrix} -0.1 & 1.4 \\ 1 & -2 \end{vmatrix} = -1.2$$

- since $|-0.99| < |-1.2|$, the system is **stable**.

Example 5

Example

The block diagram of a sampled data system is shown below. Use Jury Test to determine the value of K for which the system is stable. Assume that $K > 0$ and $T = 1$ s.



Example 5

Solution

- The characteristic equation is:

$$1 + G(z) = 0$$

$$G(z) = \mathscr{Z}\left\{\frac{1-e^{-Ts}}{s}\frac{K}{s(s+1)}\right\} = (1-z^{-1})\mathscr{Z}\left\{\frac{k}{s^2(s+1)}\right\}$$

$$= \frac{K(0.368z+0.264)}{(z-1)(z-0.368)}$$

$$z^2 - z(1.368-0.368K) + 0.368 + 0.264K = 0$$

Example 5

Solution

- Apply Jury test for 2^{nd} order equation:

$$F(z) = a_2 z^2 + a_1 z + a_0 = 0, \quad where \quad a_2 > 0$$

$$z^2 - z(1.368 - 0.368K) + 0.368 + 0.264K = 0$$

$$F(1) > 0, \quad F(-1) > 0, \quad |a_0| < a_2$$

$$F(1) = 0.632K > 0 \quad \Rightarrow \boxed{K > 0}$$

$$F(-1) = 2.736 - 0.104K > 0 \quad \Rightarrow \boxed{K < 26.3}$$

- The third condition is:

$$|a_0| < a_2$$

$$|0.368 + 0.264K| < 1$$

$$-1 < 0.368 + 0.264K < 1$$

$$-5.18 < K < 2.4$$

- Combining all inequalities together, the system is stable for $\boxed{0 < K < 2.4}$

Example 6

Example

Determine the stability of the system having the following characteristic equation:

$$F(z) = z^4 + z^3 + 2z^2 + 2z + 0.5 = 0$$

$$F(1) = 6.5 > 0, ✓$$

$$(-1)^4 F(-1) = 1 - 1 + 2 - 2 + 0.5 > 0, ✓$$

$$|a_0| = 0.5 < 1 = a_4 ✓$$

$$|b0| = 0.75 > |b3| = 1.5 ✗$$

$$|c0| = 1.6875 > |c2| = 0.75 ✓$$

- System is **unstable**!

z^0	z^1	z^2	z^3	z^4
0.5	2	2	1	1
1	1	2	2	0.5
−0.75	0	−1	**−1.5**	
−1.5	−1	0	−0.75	
−1.6875	−1.5	**0.75**		

Routh–Hurwitz Criterion

- The stability of a sampled data system can be analyzed by transforming the system characteristic equation into the s-plane and then applying the well-known Routh–Hurwitz criterion.
- A bilinear transformation is usually used to transform the interior of the unit circle in the z-plane into the left-hand s-plane (ω-plane). For this transformation, z is replaced by:

$$z = \frac{1+\omega}{1-\omega} \quad \Rightarrow \quad F(\omega) = b_n\,\omega^n + b_{n-1}\,\omega^{n-1} + \cdots + b_1\,\omega + b_0 = 0$$

Routh–Hurwitz criterion

number of roots of the characteristic equation in the right hand s-plane is equal to the number of sign changes of the coefficients in the first column of the array.

Routh–Hurwitz Criterion

- Routh-Hurwitz array is formed as:

$$
\begin{array}{l|llll}
\omega^n & b_n & b_{n-2} & b_{n-4} & \cdots \\
\omega^{n-1} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
\omega^{n-1} & c_1 & c_2 & c_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots \\
\omega^1 & j_1 & & & \\
\omega^0 & k_1 & & &
\end{array}
$$

- 1^{st} two rows are obtained from the equation directly and the other rows are calculated as:

$$
c_1 = \frac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}}, \quad c_2 = \frac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}}, \quad \cdots
$$

- Thus, for a stable system **all coefficients** in **1^{st} column** must have the **same sign**.

Example 7

- The characteristic equation of a sampled data system is given by

$$2z^3 + z^2 + z + 1 = 0$$

- Determine the stability of the system using the Routh–Hurwitz criterion.

$$2\left(\frac{1+\omega}{1-\omega}\right)^3 + \left(\frac{1+\omega}{1-\omega}\right)^2 + \left(\frac{1+\omega}{1-\omega}\right) + 1 = 0$$

$$2(1+\omega)^3 + (1-\omega)(1+\omega)^2 + (1-\omega)^2(1+\omega) + (1-\omega)^3 = 0$$

$$\omega^3 + 7\omega^2 + 3\omega + 5 = 0$$

- Now, we form Routh array:

$$\begin{array}{c|cc} \omega^3 & 1 & 3 \\ \omega^2 & 7 & 5 \\ \omega^1 & 16/7 & \\ \omega^0 & 5 & \end{array}$$

- No sign change in the first column, so the system is **stable**.

Example 7

Answer Check

- roots of the characteristic equation: $2z^3 + z^2 + z + 1 = 0$ can be found using MATLAB
- with the commands:

```
roots([2 1 1 1])
```

```
abs(roots([2 1 1 1]))
```

roots([2 1 1 1])	abs(roots([2 1 1 1]))
$0.1195 + j\,0.8138$	0.8226
$0.1195 - j\,0.8138$	0.8226
-0.7390	0.7390

- all roots are less than one, i.e. the roots lie inside unit circle. Hence, we can conclude that the system is **stable**.

Thanks for your attention.

Questions?

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https://mnourgwad.github.io

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