Digital Control

CSE421

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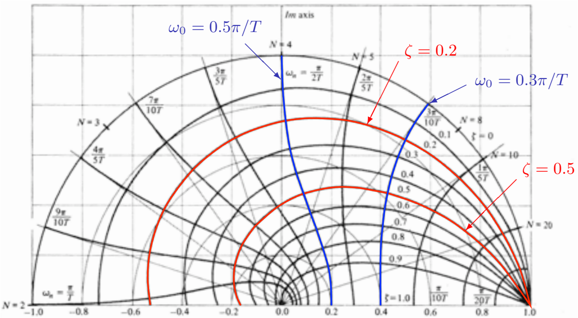
Lecture 7: **Discrete-time Block Diagrams**



Lecture: 7

Discrete-time Block Diagrams

- Derive the pulse transfer function of a continuous-time system driven by discrete input.
- Manipulate block diagrams of open and closed-loop discrete-time systems.

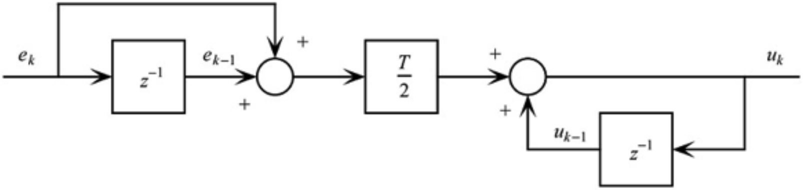
Block Diagrams of Discrete Systems

- All linear difference equations are composed of delays, multiplies, and adds, and we can represent these operations in block diagrams.
- block diagrams are often helpful in system visualization.

Example

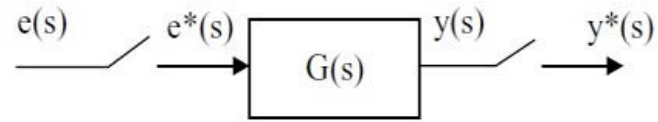
Consider the difference equation for trapezoidal integration: $u_k = u_{k-1} + \frac{T}{2}(e_k + e_{k-1})$

- This difference equation is represented by the block diagram shown.



Laplace transform of sampled signals

- Consider the following continuous-time system driven by discrete input $e^*(s)$:



- Suppose that we are interested in finding the relationship between the sampled output $y^*(s)$ and the sampled input $e^*(s)$.
- The continuous output $y(s)$ is given by:

$$y(s) = G(s)e^*(s)$$

- Sampling the output signal gives:

$$y^*(s) = [G(s)e^*(s)]^*$$

Laplace transform of sampled signals

- It was shown that a sampled signal $r^*(t)$ can be described as a multiplication of the original continuous-time signal $r(t)$ by the impulse train (If we assume that $r(t) = 0,\ t < 0$),

$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\delta(t - nT)$$

- Taking Laplace transform of both sides we get:

$$R^*(s) = \sum_{n=0}^{\infty} r(nT)e^{-nTs}$$

Note: $\mathscr{L}\{\delta(t-\tau)\} = e^{-\tau s}$

Laplace transform of sampled signals

- since $z = e^{Ts}$, then we can write:

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n}$$

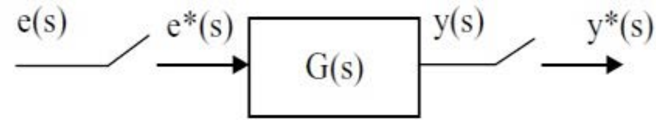
- Which is the definition of the z-transform. So, we conclude that the **z-transform of a sequence of samples is equivalent to the Laplace transform of the sampled signal** with the following substitution:

$$R(z) = R^*(s)\big|_{z=e^{Ts}}$$

- This is a useful observation that we will use next.

Pulse Transfer Function

- Consider the following continuous-time system driven by discrete input $e^*(s)$:



- Suppose that we are interested in finding the relationship between the sampled output $y^*(s)$ and the sampled input $e^*(s)$.
- The continuous output $y(s)$ is given by:

$$y(s) = G(s)e^*(s)$$

- Sampling the output signal gives:

$$y^*(s) = [G(s)e^*(s)]^*$$

Pulse Transfer Function

$$y^*(s) = [G(s)e^*(s)]^* = \frac{1}{T}\sum_{k=-\infty}^{\infty} y(s - jk\omega_s)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s - jk\omega_s)e^*(s - jk\omega_s)$$

$$e^*(s - jk\omega_s) = e^*(s)$$ (since sampling an already sampled signal has no further effect)

$$y^*(s) = e^*(s)\frac{1}{T}\sum_{k=-\infty}^{\infty} G(s - jk\omega_s) = G^*(s)e^*(s)$$

$$Y(z) = G(z)e(z)$$

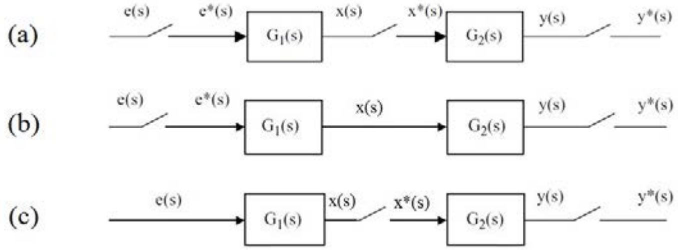
- Thus, we have the following **rule of sampling**:

$$[Ge^*]^* = G^*e^*$$

Sampler Location Effect

on Cascade System Transfer Function

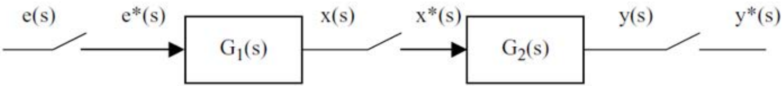
- In a discrete-time system including several analog subsystems in cascade and several samplers,
  - the location of the sampler plays an important role in determining the overall transfer function.
- Consider the following three cascade systems:



Example 1

Example

Derive an expression for the z-transform of the following system output:



- The following expressions can be written for the system:

$$X(s) = G_1(s)e^*(s), \quad \Rightarrow \quad X^*(s) = G_1^*(s)e^*(s),$$
$$Y(s) = G_2(s)X^*(s), \quad \Rightarrow \quad Y^*(s) = G_2^*(s)X^*(s),$$

- Which gives:

$$Y^*(s) = G_1^*(s)G_2^*(s)E^*(s),$$
$$Y(z) = G_1(z)G_2(z)E(z).$$

Example 1

- For example, if

$$G_1(s) = \frac{1}{s}, \quad G_2(s) = \frac{a}{s+a}$$

- Then:

$$\mathscr{Z}\{G_1(s)\} = \frac{z}{z-1}, \quad \mathscr{Z}\{G_2(s)\} = \frac{a\,z}{z-e^{-aT}}$$

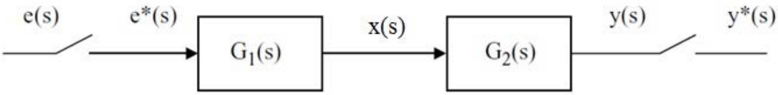
- And hence the output is given by

$$\begin{aligned} Y(z) &= \frac{z}{z-1}\frac{az}{z-e^{-aT}}E(z) \\ &= \frac{az^2}{(z-1)(z-e^{-aT})}E(z) \end{aligned}$$

Example 2

Example

Derive an expression for the z-transform of the following system output:



- Here, $y(t)$ is a function of all previous values of $x(t)$, not just the values at sampling instants. In contrast, $y^*(t)$ is a function of the values of $e(t)$ at the sampling instants only .
- Based on this observation, the following expressions can be obtained:

$$
\begin{aligned}
y(s) &= G_1(s)G_2(s)e^*(s) \\
y^*(s) &= [G_1(s)G_2(s)e^*(s)]^* \\
&= [G_1G_2]^*(s)e^*(s)
\end{aligned}
$$

$$Y(z) = G_1G_2(z)e(z),$$

Note: $G_1G_2(z) = \mathscr{Z}\{G_1(s)G_2(s)\}\} \neq G_1(z)G_2(z)$

Example 2

- For example, if

$$G_1(s) = \frac{1}{s}, \quad G_2(s) = \frac{a}{s+a}$$

- Then, from the z-transform tables,

$$\mathscr{Z}\{G_1(s)G_2(s)\} = \mathscr{Z}\left\{\frac{a}{s(s+a)}\right\} = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

- and the output is given by:

$$Y(z) = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}E(z)$$

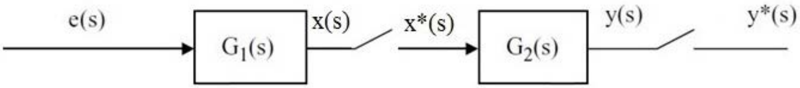
- Note that this is different from the result of Example 1:

$$Y(z) = \frac{az^2}{(z-1)(z-e^{-aT})}E(z)$$

Example 3

Example

Derive an expression for the z-transform of the output of the following open-loop sampled data system.



- The following expressions can be written for the system:

$$
\begin{aligned}
y(s) &= G_2(s)x^*(s), \quad x(s) = G_1(s)e(s) \\
x^*(s) &= [G_1e]^*(s) \Rightarrow y(s) = G_2(s)[G_1e]^*(s), \\
y^*(s) &= G_2^*(s)[G_1e]^*(s), \\
Y(z) &= G_2(z)G_1e(z),
\end{aligned}
$$

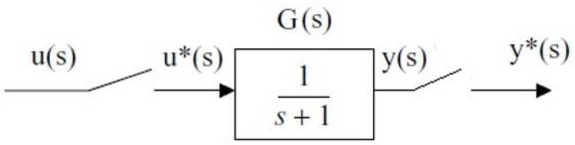
Comment:

- As seen in this example, we are not able to write a discrete-time transfer function of the system.
- In general, if the input to a sampled-data system is applied directly to a continuous-time part of the system before being sampled, the z-transform of the output cannot be expressed as a function of the z-transform of the input signal.
- However, this type of system presents no special problems in analysis and design.

Example 4

Example

A unit step signal is applied to the following system. Calculate and draw the output response of the system, assuming a sampling period of T = 1 s.



- For this system we can write:

$$y(s) = G(s)u^*(s)$$
$$y^*(s) = G^*(s)u^*(s)$$
$$Y(z) = G(z)u(z)$$

- The z-transform of a unit-step function is

$$u(z) = \frac{z}{z-1}$$

Example 4

- the z-transform of $G(s)$ is:

$$G(z) = \mathscr{Z}\{G(s)\} = \mathscr{Z}\left\{\frac{1}{s+1}\right\} = \frac{z}{z-e^{-T}}$$

- Thus, the output is given by

$$\begin{aligned} Y(z) = u(z)G(z) &= \frac{z}{z-1}\frac{z}{z-e^{-T}} \\ &= \frac{z^2}{(z-1)(z-e^{-T})}. \end{aligned}$$

- Since $T = 1$ sec and $e^{-1} = 0.368$, we get:

$$Y(z) = \frac{z^2}{(z-1)(z-0.368)}$$

- The output response can be obtained by finding the inverse z-transform of $Y(z)$. Using partial fractions,

$$\begin{aligned} \frac{Y(z)}{z} &= \frac{A}{z-1} + \frac{B}{z-0.368} \\ &= \frac{1.582}{z-1} - \frac{0.582}{z-0.368} \\ Y(z) &= \frac{1.582z}{z-1} - \frac{0.582z}{z-0.368} \end{aligned}$$

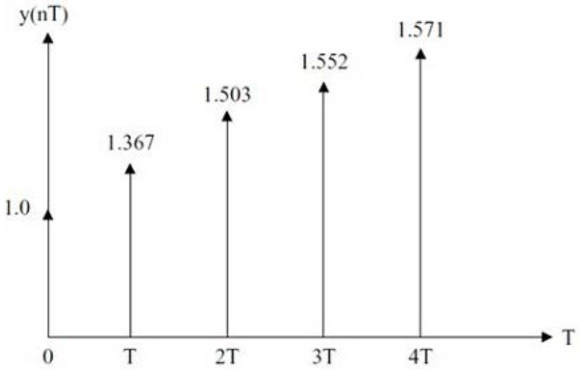
- From the z-transform tables we find

$$y(n) = 1.582 - 0.582\,(0.368)^n$$

Example 4

- The first few output samples are:

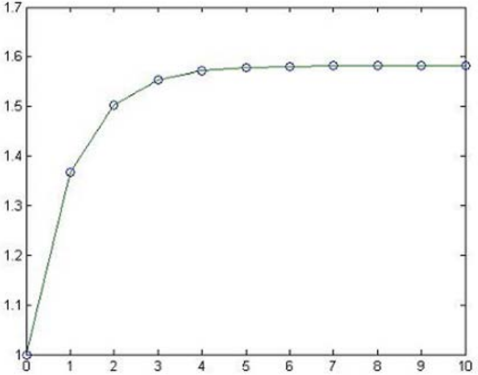
$$
\begin{aligned}
y(0) &= 1, \\
y(1) &= 1.367, \\
y(2) &= 1.503, \\
y(3) &= 1.552, \\
y(4) &= 1.571, \\
&\vdots
\end{aligned}
$$



- It is important to note that the response is only known at the sampling instants. That is the inter-sample behavior cannot be determined by the z-transform method of analysis.

Example 4

Using MATLAB




```
s = tf('s');
G = 1/(s+1);
Gd = c2d(G,1);
%zoh is the default conversion
    method for c2d!
t = 0:1:10;
y = step(Gd,t)
plot(t,y,'o',t,y)
Y=
0  0.6321  0.8647  0.9502
    0.9817  0.9933  0.9975
    0.9991  0.9997  0.9999
    1.0000
```

- Note that there is "o" at each sample point, this was determined by the argument in the plot function. The points are also connected with lines. Although this is not true for discrete output but it displays the response more clearly.

MATLAB Command dlsim

- Command `dlsim` can be used to simulate the response of discrete systems to different kinds of input signals.
- For example, to find the step response of a discrete transfer function:

$$G(z) = \frac{0.1813}{z - 0.8187}, \quad T = 0.2$$

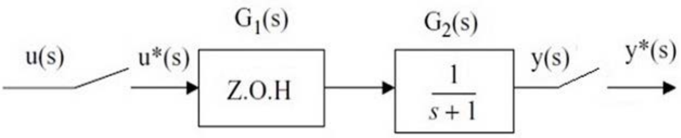
```
u = ones(size(0:10));  % Define input to be 1 from samples 0 to 10
num = [0, 0.1813];       % Numerator coefficients of z
den = [1, -0.8187];      % Denominator coefficients of z
y = dlsim(num,den,u);
plot(0:10,y,'o',0:10,y); % Plot samples as 'o' connected by lines
```

Example 5

Example

Assume that the previous system is modified so that G2(s) is preceded by a ZOH. What will the system output be if the applied input is

- a unit step and
- a unit ramp.



Example 5

Solution

- The transfer function of the ZOH is:

$$G_{ZOH}(s) = G_1(s) = \frac{1 - e^{-Ts}}{s}$$

- For this system, we can write:

$$Y(z) = G_{ZOH}G_2(z)u(z), \quad G_{ZOH}G_2(s) = \frac{1 - e^{-Ts}}{s}\frac{1}{s+1}$$

- Using partial fractional expansion we can write

$$G_{ZOH}G_2(s) = (1 - e^{-Ts})\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

- From the z-transform tables

$$G_{ZOH}G_2(z) = (1 - z^{-1})\left(\frac{z}{z-1} - \frac{z}{z-e^{-1}}\right) = \frac{0.63}{z - 0.37}$$

Example 5

Solution

- For a unit step input,

$$u(z) = \frac{z}{z-1}$$

- And the system output is given by

$$Y(z) = \frac{0.63z}{(z-1)(z-0.37)}$$

- Using partial fraction method, we can write

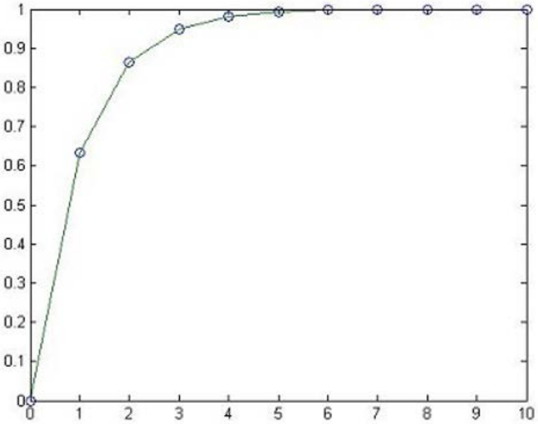
$$\frac{Y(z)}{z} = \frac{1}{z-1} - \frac{1}{z-0.37}$$

- Hence,

$$y(n) = 1 - (0.37)^n$$

Example 5

Using MATLAB




```
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G = 1/(s+1);
Gd = c2d(G,1);  % zoh is the default
      conversion method for c2d!
t = 0:1:10;
y = step(Gd,t)
plot(t,y,'o',t,y)

Y=
    0
    0.6321
    0.8647
    0.9502
    0.9817
    0.9933
    0.9975
    0.9991
    0.9997
    0.9999
    1.0000
```

Example 5

unit Ramp Input

- For a unit ramp input:

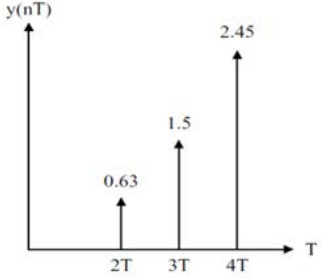
$$u(z) = \frac{Tz}{(z-1)^2}$$

- And the system response (with $T = 1$) is given by:

$$Y(z) = \frac{0.63z}{(z-1)^2(z-0.37)} = \frac{0.63z}{z^3 - 2.37z^2 + 1.74z - 0.37}$$

- Using long division, we obtain the first few output samples as:

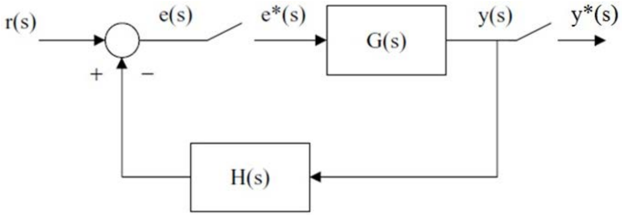
$$Y(z) = 0.63z^{-2} + 1.5z^{-3} + 2.45z^{-4} + 3.43z^{-5} + \cdots$$



Example 6

Example

Derive an expression for the transfer function of the closed-loop sampled data system:



- **Solution**

$$y(s) = G(s)e^*(s),$$

$$e(s) = r(s) - H(s)y(s)$$

$$e(s) = r(s) - H(s)G(s)e^*(s)$$

$$e^*(s) = r^*(s) - [H(s)G(s)]^* e^*(s)$$

$$e^*(s) = \frac{1}{1 + [H(s)G(s)]^*} r^*(s)$$

$$y(s) = \frac{G(s)}{1 + [H(s)G(s)]^*} r^*(s)$$

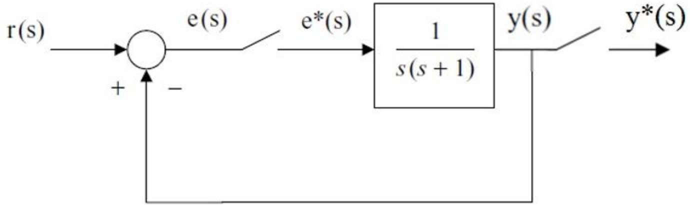
$$y^*(s) = \frac{G^*(s)}{1 + [H(s)G(s)]^*} r^*(s)$$

$$Y(z) = \frac{G(z)}{1 + GH(z)} R(z)$$

Example 7

Example

A unit step signal is applied to the sampled data system in the previous example. Calculate and plot the output response of the system. Assume that $T = 1$ s.



Example 7

Solution

- The response of the previous system was found to be:

$$Y(z) = \frac{G(z)}{1 + GH(z)} r(z)$$

$$R(z) = \frac{z}{z-1}$$

$$GH(s) = \frac{1}{s(s+1)}$$

$$GH(z) = \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$$

- Substituting in $Y(z)$ gives:

$$Y(z) = \frac{\frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}}{1 + \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}} \frac{z}{z-1}$$

$$= \frac{z^2(1-e^{-T})}{(z-1)(z^2 - 2e^{-T}z + e^{-T})}.$$

- Since $T = 1$:

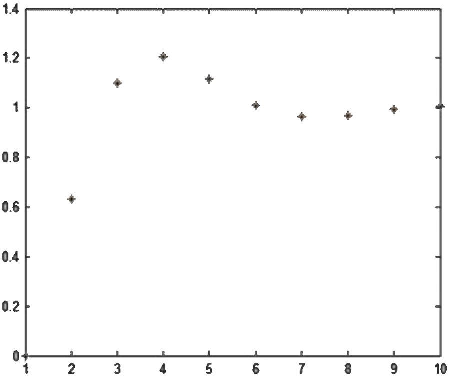
$$Y(z) = \frac{0.632z^2}{z^3 - 1.736z^2 + 1.104z - 0.368}$$

Solution

- Through long division, we obtain the first few terms:

$$Y(z) = 0.632z^{-1} + 1.096z^{-2} + 1.25z^{-3} + \cdots$$

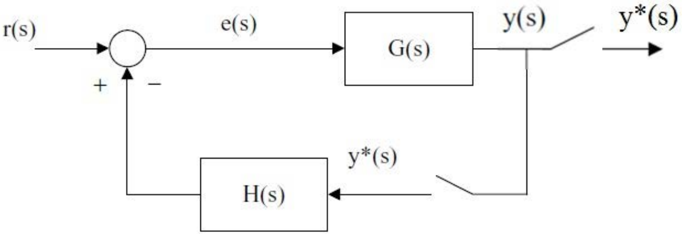
- The first 10 samples of the response are shown next:



Example 8

Example

Derive an expression for the output function of the following closed-loop sampled data system.



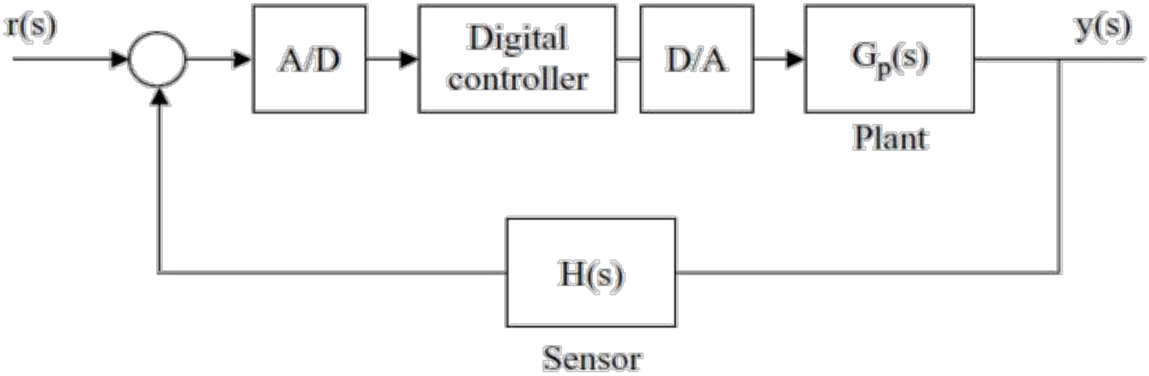
- **Solution**:

$$Y(z) = \frac{G(z)}{1 + GH(z)} R(z)$$

Example 9

Example

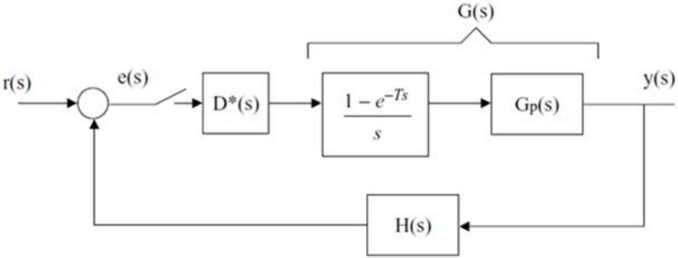
Derive an expression for the output function of the following closed-loop sampled data system.



Example 9

Solution

- The **ADC** can be approximated with an **ideal sampler** and **DAC** with a **ZOH**.
- Denoting the digital controller by D(z) and
- Combining ZOH and plant into G(s), the system block diagram is combined to be:

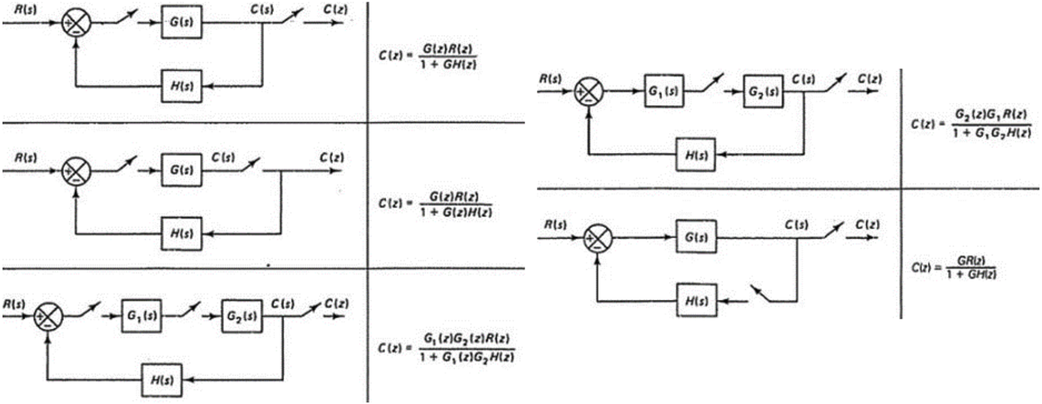


- Now, try to **verify** that the corresponding transfer function is:

$$\frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)GH(z)}$$

Summery

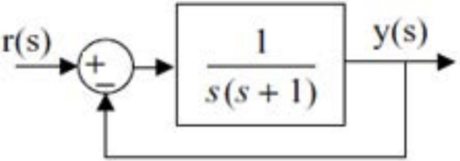
Five typical configurations for closed loop discrete time control systems



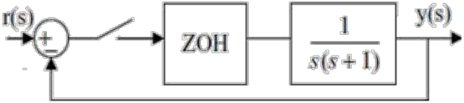
Continuous vs. discrete-time control systems

a comparison

- Consider the next continuous-time closed-loop system.



- We add a sampler (ADC) and a zero-order hold (DAC) to form its **equivalent** discrete-time system.



- Then we derive equations for the step responses of both systems, plot and compare them.

Continuous-time system response

- The transfer function of the continuous closed-loop system is:

$$\frac{y(s)}{r(s)} = \frac{G(s)}{1+G(s)} = \frac{1}{s^2+s+1}$$

- Since the input is unit step, r(s)=1/s, the output becomes:

$$y(s) = \frac{1}{s(s^2+s+1)}$$

- To find the inverse Laplace transform, we write:

$$y(s) = \frac{1}{s} - \frac{s+1}{s^2+s+1} = \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+(0.866)^2} - \frac{0.5}{(s+0.5)^2+(0.866)^2}$$

- From which we find the time response:

$$y(t) = 1 - e^{-0.5t}\left[\cos(0.866t) + 0.573\sin(0.866t)\right]$$

Discrete-time system response

- The transfer function of the previous discrete-time system is:

$$\frac{Y(z)}{r(z)} = \frac{G(z)}{1+G(z)}, \quad \text{and } r(z) = \frac{z}{z-1}$$

- the z-transform of the plant is given by:

$$G(s) = \frac{1-e^{-Ts}}{s^2(s+1)}$$

- Expanding by means of partial fractions, we obtain:

$$G(s) = (1-e^{-Ts})\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)$$

Discrete-time system response

- From z-transform tables we obtain:

$$G(z) = (1 - z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

- Setting $T = 1$ s and simplifying gives:

$$G(z) = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

- Therefore, The transfer function of the closed loop discrete-time system is:

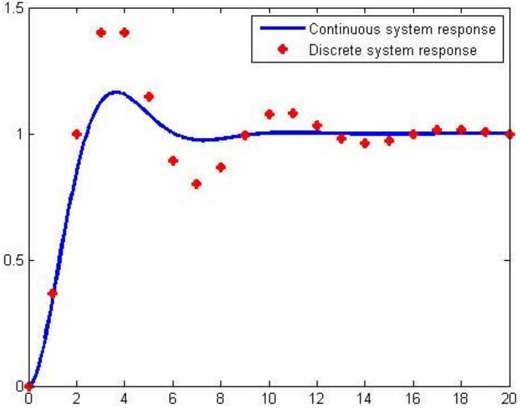
$$\frac{Y(z)}{r(z)} = \frac{G(z)}{1+G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632} \quad \Rightarrow \quad Y(z) = \frac{z(0.368z + 0.264)}{(z-1)(z^2 - z + 0.632)}$$

- The inverse z-transform can be found by long division:

$$Y(z) = 0.368z^{-1} + z^{-2} + 1.4z^{-3} + 1.4z^{-4} + 1.15z^{-5} + 0.9z^{-6} + \cdots$$

Discrete-time system response

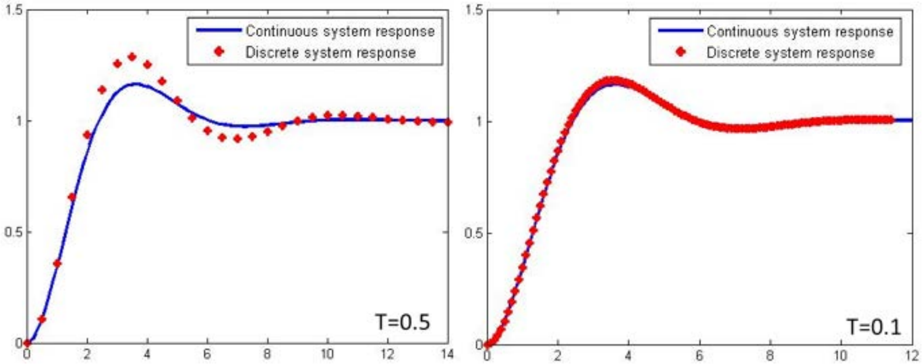
- Time response of both continuous-time and discrete-time systems:



- It is clear that the **discrete system has higher overshoot**.
- This indicates that **sampling process has a destabilizing effect** on the system.

Discrete-time system response

- Let us now decrease the sampling interval to 0.5 and 0.1, respectively, and repeat the comparison.



- It is clear that the response of the discrete system **approaches** that of the continuous response as T becomes **sufficiently small**.

Discrete-time system response

MATLAB code

```
s = tf('s');
t=0:0.1:35;
% continuous system
G = 1/(s^2+s);
Gcl = G/(1+G)
y=step(Gcl,t)
plot(t,y,'LineWidth',2.5)
hold on;
% discrete system
% choose the sampling interval
T = 0.1;
Gd = c2d(G,T);
Gcld = Gd/(1+Gd)
[y,t1]=step(Gcld)
plot(t1,y,'*r','LineWidth',2.5)
axis([0 12 0 1.5])
```

```
Gcl =
s^2 + s
----------------------
s^4 + 2 s^3 + 2 s^2 + s

Continuous-time transfer function.

Gcld =
0.004837 z^3 - 0.004536 z^2 - 0.004535 z + 0.004234
---------------------------------------------------
z^4 - 3.805 z^3 + 5.434 z^2 - 3.452 z + 0.823

Sample time: 0.1 seconds
Discrete-time transfer function.
```

Choice of the sampling interval

- a **suitable** sampling interval must be chosen.
- Choosing a **large** sampling time has **destabilizing** effects on the system
  - as the controller has no information about what is going on for the process output between samples.
- from practical applications in the process industry,
  - sampling interval of 1 second is generally short enough for most applications (e.g, pressure, temperature and flow control).
  - Systems with fast responses (electromechanical systems, e.g. motors) require much shorter sampling intervals, usually of the order of milliseconds.

Choice of the sampling interval

- Various empirical rules have been suggested by many researchers for the selection of sampling interval.
  - These rules are based on **practical** experience and **simulation** results.

Empirical rules for the selection of sampling interval:

- $T < \tau/10$, where τ is the plant **dominant time constant**
- $T < T_{ss}/10$, where T_{ss} is the closed-loop system **settling time**

Thanks for your attention.

Questions?

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