Digital Control

CSE421

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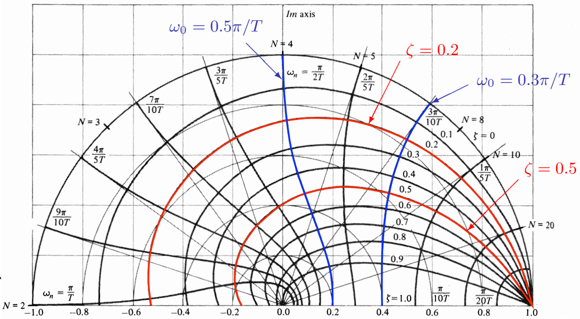
Lecture 6: **Sampling and Aliasing**

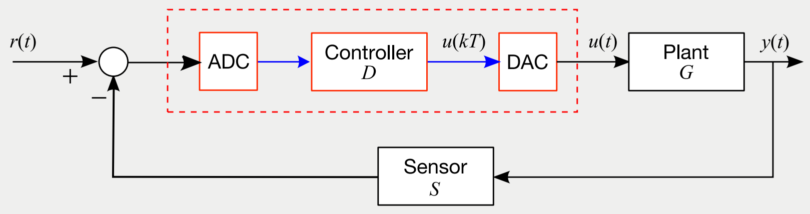


Lecture: 6

Sampling and Aliasing

- Describe mathematically the impulse sampling process.
- Recognize the frequency spectrum of a sampled signal.
- Identify aliasing phenomena.
- ideally recovering a continuous time signal from its sampled version
- Identify the disadvantages of ideal signal reconstruction.
- Describe ZOH as a simpler and effective reconstruction operation.

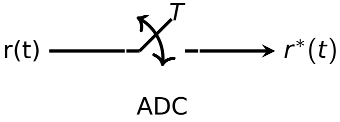
Introduction

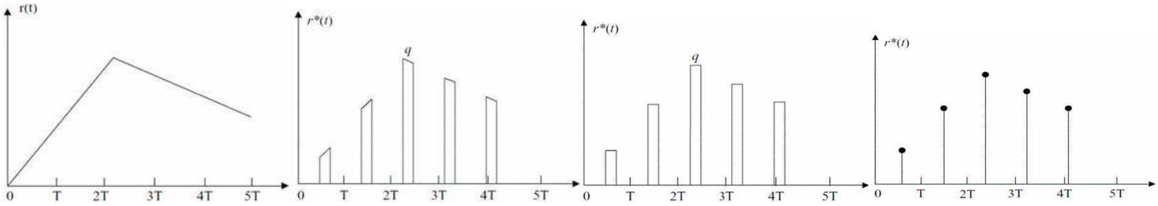


- A digital or sampled-data control system operates on discrete-time rather than continuous-time signals.
- **Due to the sampling process, some new phenomena appear** which need careful investigation. This is discussed next.

The sampling process

- A sampler is basically a **switch** that closes every T seconds.
- Where q is the amount of **time the switch is closed**
- In practice, $q \ll T$,
- the pulses can be approximated by **flat-topped** rectangles.
- If q is neglected, the operation is called **ideal sampling**





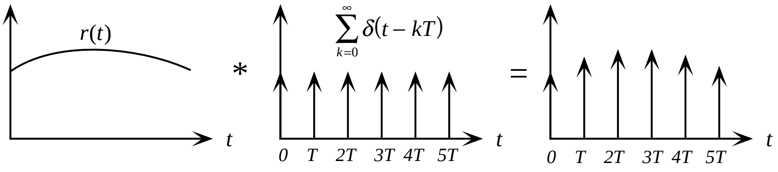
Ideal sampling

- Ideal sampling of a continuous signal can be considered as a multiplication of the signal, $r(t)$, with an impulse train $P(t)$
- an impulse train $P(t)$ is defined as:

$$P(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

- Thus, the sampled signal $r^*(t)$ is:

$$r^*(t) = P(t)\, r(t) = r(t) \sum_{-\infty}^{\infty} \delta(t - nT) = \sum_{-\infty}^{\infty} r(nT)\, \delta(t - nT)$$



Sampling of continuous time signals

- After sampling a continuous time signal $r(t)$, with sampling period T,
  - we get a sequence of samples $r(kT)$.
- The question now is: **Do we loose anything by sampling?**
  - Or is it possible to recover $r(t)$ from $r(kT)$?
- It seems that very little is lost if the sampling period T is small or the sampling frequency $\omega_s = 2\pi/T$ is high.

The question is how small (high) is enough?

- To answer this question we better study the frequency spectrum of the sampled signal $r^*(t)$.

Frequency spectrum of sampled signals

- Mathematically, a sampled signal $r^*(t)$ of the continuous $r(t)$ is given as:

$$r^*(t) = P(t)\, r(t) = r(t) \sum_{-\infty}^{\infty} \delta(t - k\,T)$$

- We will assume that the frequency spectrum of $r(t)$ is $R(\omega)$ and it is **band limited** to $[-\omega, \omega]$.

$$r(t) \Leftrightarrow R(\omega)$$

- The Fourier transform of the impulse train is also an impulse train scaled by $1/T$ where $\omega_s = 2\pi/T$.

$$\sum_{k=-\infty}^{\infty} \delta(t - \; T) \Leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\,\omega_s)$$

Frequency spectrum of sampled signals

Frequency spectrum of $r^*(t)$ is the result of multiplying $r(t)$ by impulse train:

$$r^*(t) = r(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$\text{multiplication in time is convolution in frequency} \Leftrightarrow R(\omega) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\,\omega_s)$$

$$\text{convolution of sum is sum of convolutions} \Leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} R(\omega) * \delta(\omega - k\,\omega_s)$$

$$\text{convolution is integral} \Leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} R(\tau)\,\delta(\omega - \tau - k\,\omega_s)$$

$$\text{apply shifting property of } \delta \text{ function} \Leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} R(\omega - k\,\omega_s)$$

Frequency spectrum of sampled signals

This is an important result:

$$r^*(t) \Leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} R(\omega - k\,\omega_s)$$

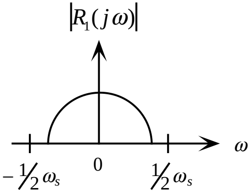
it indicates that **two things** happen to the frequency spectrum of $r(t)$ when sampled to $r^*(t)$:

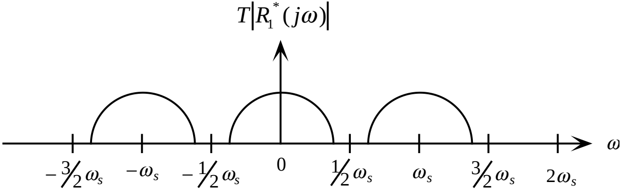
1. The **magnitude** of the sampled spectrum is **scaled** by $1/T$ of the continuous spectrum,
2. The summation indicates that there are an **infinite number of repeated spectra** in the sampled signal, and they are repeated every $\omega_s = 2\pi/T$ along the frequency axis:

$$f_a = \pm k\, f_s \pm f_o \text{ Hz}$$

Example

Consider a continuous function $r_1(t)$ which has no frequency content above half the sampling frequency $\omega_s/2$. Show the original amplitude $|R_1(j\omega)|$ and sampled $|R_1^*(j\omega)|$ spectra.

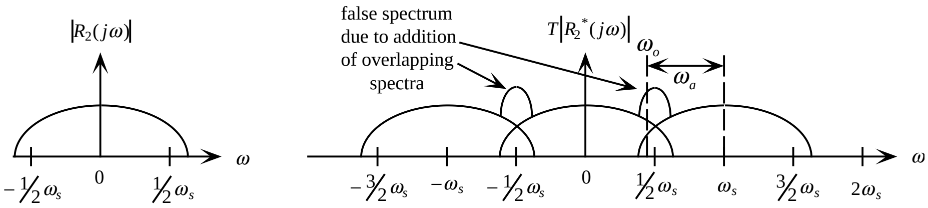




- The spectrum of $r^*(t)$ is scaled by $1/T$ and repeated (**non-overlapping**),
- it is **possible to reconstruct** $r(t)$ by extracting the original single spectrum from the array of repeated spectra (Remember that all information present in $r(t)$ is also present in original spectrum $R(j\omega)$.

Example

Now consider $r_2(t)$which has frequency content above half the sampling frequency $\omega_s/2$. Show the original amplitude $|R_2(j\omega)|$ and sampled $|R_2^*(j\omega)|$ spectra.



- The same scaling and repetition occurs in the sampled spectrum, but this time due to the addition of the **overlapping** spectra there is **no chance of recovering** the original spectrum after sampling.

Meaning of a repeated spectrum

Example

Example

Consider a signal of frequency 100 Hz, sampled at a rate of $f_s = 500$ Hz. What are the frequencies that appear at the sampled signal? Can the original signal be reconstructed from its sampled version? Why?

- The signal with $f_o = 100$ sampled at $f_s = 500$ will show up at frequencies:

$$f_a = \pm k\, f_s \pm f_o \text{ Hz}$$

$$100 \text{ Hz}, 400 \text{ Hz}, 600 \text{ Hz}, 900 \text{ Hz}, 1100 \text{ Hz}, \cdots$$

- This means that any sinusoid signal of these frequencies can pass through the samples.
- The reconstruction of the continuous time signal from discrete samples as we will see later uses some form of low-pass filtering.
- Therefore, in our case, the LPF filter will pass the 100 Hz component, which is the original signal (so, no problem).

Aliasing Effect

Example

Consider a signal of frequency 350 Hz, sampled at a rate of $f_s = 500$ Hz. What are the frequencies that appear at the sampled signal? Can the original signal be reconstructed from its sampled version? Why?

- The signal with $f_o = 100$ sampled at $f_s = 500$ will show up at frequencies:

$$f_a = \pm k\, f_s \pm f_o \text{ Hz}$$

$$1150 \text{ Hz}, 350 \text{ Hz}, 650 \text{ Hz}, 850 \text{ Hz}, 1350 \text{ Hz}, \cdots$$

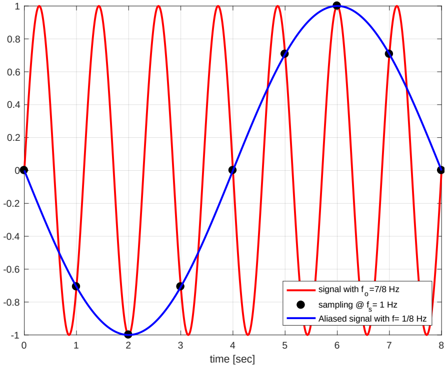
- In this case, the LPF filter will pass the 150Hz component, which is not the original signal of 350 Hz (a problem called **aliasing**).

Aliasing Effect

The impersonation of high-frequency continuous sinusoids by low-frequency discrete sinusoids, due to an insufficient number of samples in a cycle (the sampling interval is not short enough).

Meaning of a repeated spectrum

- The following two sinusoids have identical samples, and we cannot distinguish between them from their samples:



- Note that the frequency of the two sinusoids are 7/8 Hz, 1/8 Hz and sampling frequency $f_s = 1$ Hz (try to deduce these from the plot). Do you realize any pattern?

Aliasing Effect

Wagon-wheel effect, https://www.youtube.com/watch?v=jHS9JGkEOmA

- A good description: Aliasing and Nyquist - Introduction & Examples, https://www.youtube.com/watch?v=v7gjeUFxVw0

Aliasing Effect

Effect on images

Scientific Volume Imaging,https://svi.nl/wikiimg/StFargeaux_kasteel_buiten1_aliased.jpg

Solving the Aliasing Problem

- First, **sample at least twice the largest frequency in the signal** of interest.
- If the signal of interest contains noise (unwanted signal usually of high frequency),
  - it is essential that an **anti-aliasing analog filter** be used, before sampling, to filter out those frequencies above one-half the sampling frequency (called the **Nyquist frequency**).
  - Otherwise, those unwanted frequencies will erroneously appear as lower frequencies after sampling.

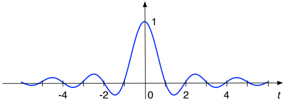
The Sampling Theorem

Sampling Theorem

A continuous time signal with a Fourier transform that is zero outside the interval $(-\omega_0, \omega_0)$, i.e. **band-limited**, can be recovered uniquely by its values in equi-distant points if the sampling frequency is higher than $2\omega_0$ (**no aliasing**)

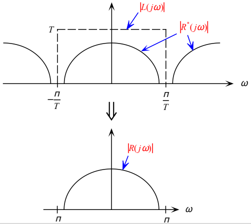
- the signal is given by:

$$r(t) = \sum_{k=-\infty}^{\infty} r(kT)\,\mathrm{sinc}\,\frac{\pi(t-kT)}{T}$$



- This equivalent to passing $R^*(j\omega)$ through an **ideal low pass filter** $L(j\omega)$, with magnitude:

$$|L(j\omega)| = \begin{cases} T, & -\frac{\pi}{T} \leq \omega \leq \frac{\pi}{T} \\ 0, & \omega > \frac{\pi}{T} \end{cases}$$

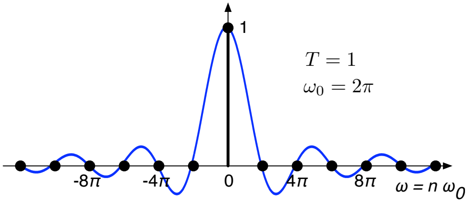


- The filter passes the original spectrum while rejecting higher frequency components

Comments on Sampling Theorem

- The reconstruction equation

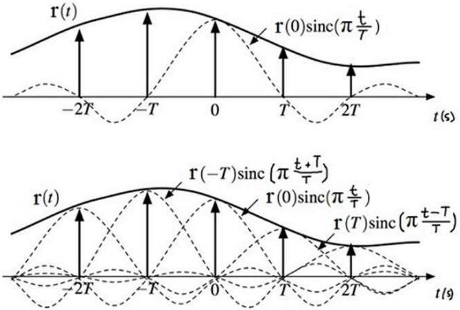
$$r(t) = \sum_{k=-\infty}^{\infty} r(kT)\,\mathrm{sinc}\,\frac{\pi(t-kT)}{T}$$



- shows how to reconstruct function $r(t)$ (band-limited,no aliasing!) from its samples $r^*(t)$.
  - `sinc()` function fills in the gaps between samples.
  - However, from ideal LPF impulse response: $l(t) = sinc\frac{\pi t}{T}$
  - Since this is the response due to an impulse applied at $t = 0$, the **ideal** reconstruction filter is **non-causal**, because its response begins before it receives the input. This means we can not apply the filter in real time.

Comments on Sampling Theorem

- The interpolated signal is a **sum of shifted** `sinc`s, **weighted** by the samples $r(kT)$.



- The `sinc` function $h(t) = sinc\pi\, t/T$ shifted to kT, i.e.

$$h(t-kT) = \begin{cases} 1 & \text{at } kT \\ 0 & \text{at } mT, m \neq k \end{cases}$$

- The sum of the weighted shifted `sinc`s will agree with all samples $r(kT)$.

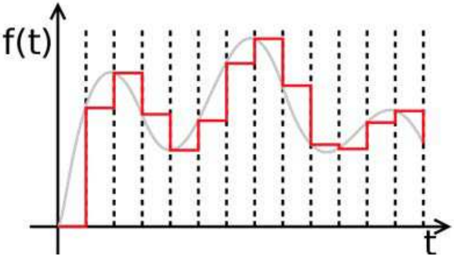
Zero-Order Hold as a Reconstruction Filter

- **Ideal LPF filter**, being non-causal, **cannot be used in real time**, and furthermore it is too complicated.
- So we will examine the behavior of a zero-order hold as a way to reconstruct continuous signal from discrete samples.

- The ZOH remembers the last information until a new sample is obtained, i.e. it takes the value $r(kT)$ and holds it constant for $kT \leq t < (k+1)T$.

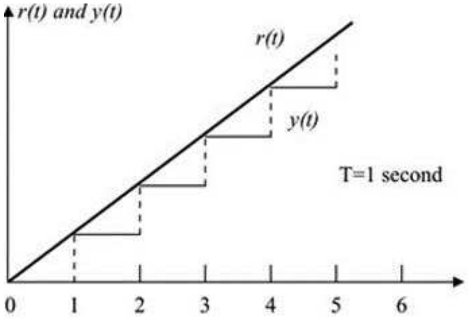
$$r(t) = r(kT), \quad kT \leq t < (k+1)T$$

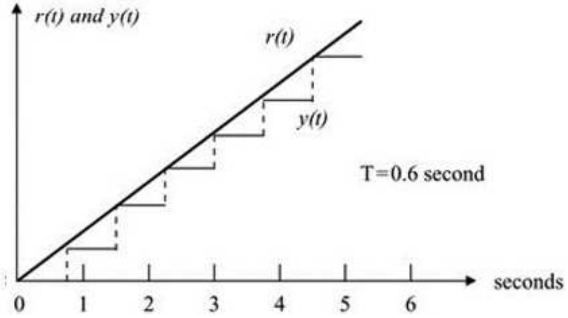
- This is exactly the behavior of a DAC in converting a sampled signal $r^*(t)$ into continuous $r(t)$.



ZOH and Sampling Period

- A sampler and ZOH can accurately follow the input signal if the sampling time T is small compared to the transient changes in the signal.





Frequency Response of ZOH

- From the shown impulse response of ZOH, its transfer function is:

$$G_{ZOH}(s) = \frac{1}{s} - \frac{e^{-Ts}}{s} = \frac{1 - e^{-Ts}}{s}$$

- The frequency behavior of $G_{ZOH}(s)$ is $G_{ZOH}(j\omega)$,

$$G_{ZOH}(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega}$$

- Multiplying numerator and denominator by $e^{j\omega T/2}$, we get:

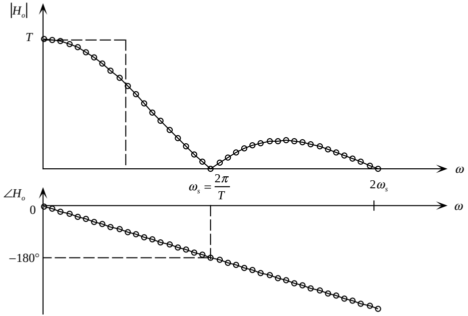
$$G_{ZOH}(s) = \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{j\omega \, e^{j\omega T/2}} = 2\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2\, j\omega \, e^{j\omega T/2}} = T\frac{\sin(\omega T/2)}{e^{j\omega T/2}\,(\omega T/2)}$$

$$= e^{-j\omega T/2}\, T \operatorname{sinc}\frac{\omega T}{2} = \left| T \operatorname{sinc}\frac{\omega T}{2} \right| \angle -(\omega T/2)$$

Frequency Response of ZOH

- The ZOH is a low-pass filter, at least an **approximation of the ideal reconstructing filter**, and has linear phase lag with frequency.

- This phase lag can be viewed as the destabilizing effect of information loss at low sampling frequencies.
- The DC magnitude of T of the ZOH compensates for the frequency scaling of $1/T$ incurred by sampling.



Thanks for your attention.

Questions?

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