CSE421

Digital Control

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Lecture 4: The z-Transform

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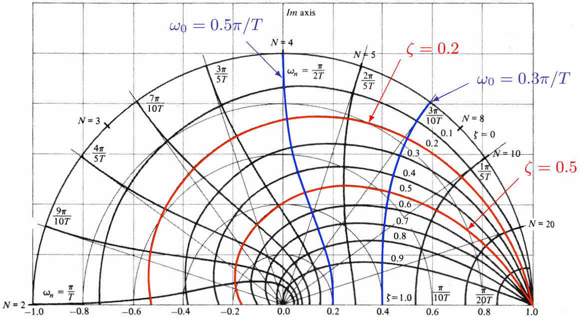


Lecture 4

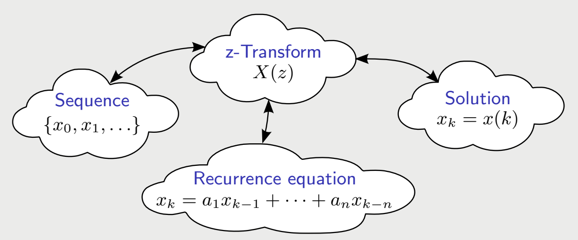
The z-Transform

- Conversion between Laplace and z-Transforms
- Some of the properties of the z-transform are:
  - Linearity and Time Shift
  - z-differentiation
  - Final value theorem
  - DC Gain of Transfer Function
- Inverse z-Transform

The z-transform

- to find $X(z)$ from $x(kT)$, the z-transform is defined as:

$$\mathscr{Z}\{x(kT)\} = \mathscr{Z}\{x_k\} = X(z)$$
$$= \sum_{k=0}^{\infty} x(kT) z^{-k} = \sum_{k=0}^{\infty} x_k z^{-k}$$
$$= x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + \cdots$$



- Discrete transfer functions are defined using z^{-1} **delay operator**
- The transfer function of a system is the z-transform of its pulse response
- $X(z)$ provides an easy way to convert between sequences, recurrence eqs and their closed-form solutions.

Tables of Laplace and z-Transforms, and z-Transform Properties

No.	Continuous Time	Laplace Transform	Discrete Time	z-Transform	
1	$\delta(t)$	1	$\delta(k)$	1	
2	$1(t)$	$\frac{1}{s}$	$1(k)$	$\frac{z}{z-1}$	
3	t	$\frac{1}{s^2}$	kT	$\frac{zT}{(z-1)^2}$	sampling t gives kT, $z\{kT\} = T\,z\{k\}$
4	t^2	$\frac{2!}{s^3}$	$(kT)^2$	$\frac{z(z+1)T^2}{(z-1)^3}$	
5	t^3	$\frac{3!}{s^4}$	$(kT)^3$	$\frac{z(z^2+4z+1)T^3}{(z-1)^4}$	
6	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	a^k	$\frac{z}{z-a}$	by setting $a = e^{-\alpha T}$.
7	$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$	$1-a^k$	$\frac{(1-a)z}{(z-1)(z-a)}$	
8	$e^{-\alpha t}-e^{-\beta t}$	$\frac{\beta-\alpha}{(s+\alpha)(s+\beta)}$	a^k-b^k	$\frac{(a-b)z}{(z-a)(z-b)}$	
9	$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$	kTa^k	$\frac{az\,T}{(z-a)^2}$	
10	$\sin(\omega_n t)$	$\frac{\omega_n}{s^2+\omega_n^2}$	$\sin(\omega_n kT)$	$\frac{\sin(\omega_n T)z}{z^2-2\cos(\omega_n T)z+1}$	
11	$\cos(\omega_n t)$	$\frac{s}{s^2+\omega_n^2}$	$\cos(\omega_n kT)$	$\frac{z[z-\cos(\omega_n T)]}{z^2-2\cos(\omega_n T)z+1}$	
12	$e^{-\zeta\omega_n t}\sin(\omega_d t)$	$\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$	$e^{-\zeta\omega_n kT}\sin(\omega_d kT)$	$\frac{e^{-\zeta\omega_n T}\sin(\omega_d T)z}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$	
13	$e^{-\zeta\omega_n t}\cos(\omega_d t)$	$\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$	$e^{-\zeta\omega_n kT}\cos(\omega_d kT)$	$\frac{z[z-e^{-\zeta\omega_n T}\cos(\omega_d T)]}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$	
14	$\sinh(\beta t)$	$\frac{\beta}{s^2-\beta^2}$	$\sinh(\beta kT)$	$\frac{\sinh(\beta T)z}{z^2-2\cosh(\beta T)z+1}$	
15	$\cosh(\beta t)$	$\frac{s}{s^2-\beta^2}$	$\cosh(\beta kT)$	$\frac{z[z-\cosh(\beta T)]}{z^2-2\cosh(\beta T)z+1}$	

No.	Property	Formula
1	Linearity	$\mathscr{Z}\{\alpha f_1(k)+\beta f_2(k)\} = \alpha F_1(z)+\beta F_2(z)$
2	Time Delay	$\mathscr{Z}\{f(k-n)\} = z^{-n}F(z)$
3	Time Advance	$\mathscr{Z}\{f(k+1)\} = zF(z)-zf(0)$ $\mathscr{Z}\{f(k+n)\} = z^nF(z)-z^nf(0)-z^{n-1}f(1)\cdots-zf(n-1)$
4	Discrete-Time Convolution	$\mathscr{Z}\{f_1(k)*f_2(k)\} = \mathscr{Z}\left\{\sum_{i=0}^{k} f_1(i)f_2(k-i)\right\} = F_1(z)F_2(z)$
5	Multiplication by Exponential	$\mathscr{Z}\{a^{-k}f(k)\} = F(az)$
6	Complex Differentiation	$\mathscr{Z}\{k^m f(k)\} = \left(-z\frac{d}{dz}\right)^m F(z)$
7	Final Value Theorem	$f(\infty) = \lim_{k\to\infty} f(k) = \lim_{z\to1}(1-z^{-1})F(z) = \lim_{z\to1}(z-1)F(z)$
8	Initial Value Theorem	$f(0) = \lim_{k\to0} f(k) = \lim_{z\to\infty} F(z)$

Properties of z-Transform

Linearity and Time shift

- **Linearity**: $\mathscr{Z}\{\alpha\, f(k) \pm \beta\, g(k)\} = \alpha\, \mathscr{Z}\{f(k)\} \pm \beta\, \mathscr{Z}\{g(k)\}$
- **Time Delay**: $\mathscr{Z}\{f(k-n)\} = z^{-n}F(z)$
- **Time Advance**: $\mathscr{Z}\{f(k+n)\} = z^{n}F(z) + \sum_{i=0}^{n-1} f(i) z^{n-i}$

Example

Obtain closed form z-transform of the sequence: $\{0, 1, 2, 4, 0, 0, \cdots\}$ using the table of z-transforms, linearity and time delay properties.

- The sequence can be written in terms of transforms of standard functions:

$$\{0, 1, 2, 4, 0, 0, \cdots\} = \{0, 1, 2, 4, 8, 16, \cdots\} - \{0, 0, 0, 0, 8, 16, \cdots\} = f(k) - g(k)$$

$$\text{where } f(k) = \begin{cases} 2^{k-1} & k > 0, \\ 0 & k \le 0 \end{cases} \qquad g(k) = \begin{cases} 8 \times 2^{k-4} & k > 4, \\ 0 & k \le 4 \end{cases}$$

$$\mathscr{Z}\{0, 1, 2, 4, 0, 0, ...\} = z^{-1}\frac{z}{z-2} - z^{-4}\frac{8z}{z-2} = \frac{z^3 - 8}{z^3(z-2)}$$

- **Multiplication by k**: $\mathscr{Z}\{k^m f(k)\} = \left(-z\frac{d}{dz}\right)^m F(z)$

Example

if $F(z) = \mathscr{Z}\{f(n)\} = \mathscr{Z}\{2^n\} = \dfrac{z}{z-2}$, use the complex differentiation property to find $G(z)$ for $g(k) = n\,2^n$

$$f(n) = 2^n \Leftrightarrow F(z) = \frac{z}{z-2}$$
$$g(n) = n\,2^n$$
$$G(z) = -z\frac{d}{dz}F(z) = \frac{2z}{(z-2)^2}.$$

Properties of z-Transform

Final Value Theorem

- **final value of the time response**: $f(\infty) = \lim_{n\to\infty} f(n) = \lim_{z\to 1}(1 - z^{-1})\, F(z)$
- this theorem is **valid only if the system is stable** (poles of $F(z)$ inside or on the unit circle i.e. the system reaches a final value).

Example

Find the final value of g(n), if $G(z) = \dfrac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)}$,

- Using the final value theorem,

$$\begin{aligned} g_\infty = \lim_{n\to\infty} g(n) &= \lim_{z\to 1}(1 - z^{-1})G(z) \\ &= \lim_{z\to 1}(1 - z^{-1})\frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)}, \\ &= \lim_{z\to 1}\frac{0.792}{(z^2 - 0.416z + 0.208)} = 1. \end{aligned}$$

Properties of z-Transform

DC Gain of Transfer Function

- For the transfer function $H(z) = \dfrac{Y(z)}{U(z)}$ is $\frac{y_\infty}{u_\infty} = H(1)$
- Let input $u(k)$ be a step of magnitude u_∞, with z-transform

$$U(z) = \frac{u_\infty\, z}{z-1}$$

- The output is given by:

$$Y(z) = H(z)\, U(z) = H(z) \frac{u_\infty\, z}{z-1}$$

- The final value of the output y(k) can be found using the final value theorem:

$$y_\infty = \lim_{k\to\infty} y_k = \lim_{z\to 1}(1 - z^{-1})\, Y(z) = \lim_{z\to 1}(1 - z^{-1})\, H(z) \frac{u_\infty\, z}{z-1} = u_\infty H(1)$$

- Hence the DC gain of the transfer function H(z) is:

$$\frac{y_\infty}{u_\infty} = H(1)$$

- Again, note that when finding the DC gain of a transfer function, all poles of the transfer function must be inside the unit circle.

Properties of z-Transform

DC Gain of Transfer Function

Example

Consider the transfer function given by

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z+1}{z^2 - 0.5z + 0.5} = \frac{z+1}{(z - 0.25 + j0.66)(z - 0.25 - j0.66)}$$

- first, it is necessary to check system stability
  - The poles are $z_{1,2} = 0.25 \pm j0.66$ then $|z_{1,2}| = 0.7058 < 1$ which means the **system is stable**.
- The DC gain is given by

$$H(1) = \frac{1+1}{1 - 0.5 + 0.5} = 2$$

- Thus if this discrete system were given an input that eventually reached a constant value, the output would eventually reach **twice** that value.
- If the denominator polynomial above were $z^2 - 0.5\,z + 2$,
  - the DC gain would evaluate to $H(1) = 0.8$,
- but that is **meaningless since the system is unstable** (the roots are outside the unit circle).

Conversion between Laplace and z-Transforms

Given a function $G(s)$, find $G(z)$ which denotes the z-transform equivalent of $G(s)$.

- It is important to realize that $G(z)$ is not obtained by simply substituting z for s in $G(s)$!
- **Method 1**: inverse Laplace transform then apply z-transform to the time function.
- **Method 2**: using Laplace to z-transform table
- **Method 3**: approximation

Example

Given $G(s) = \dfrac{1}{s^2 + 5s + 6}$, determine $G(z)$.

- Using **partial fraction**

$$G(s) = \frac{1}{s^2 + 5s + 6} = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$$

- Inverse Laplace transform

$$g(t) = \mathscr{L}^{-1}\{G(s)\} = e^{-2t} - e^{-3t}$$

- Substitute $t = kT$ gives:

$$g(kT) = e^{-2kT} - e^{-3kT}$$

- Finally,

$$G(z) = \frac{z}{z - e^{-2T}} - \frac{z}{z - e^{-3T}} = \frac{z\left(e^{-2T} - e^{-3T}\right)}{\left(z - e^{-2T}\right)\left(z - e^{-3T}\right)}$$

Conversion between Laplace and z-Transforms

Method 2

- From conversion table:

Laplace Transform	z-transform
$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$

- So,

$$G(s) = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$$

$$G(z) = \frac{z}{z-e^{-2T}} - \frac{z}{z-e^{-3T}}$$

$$= \frac{z(e^{-2T}-e^{-3T})}{(z-e^{-2T})(z-e^{-3T})}$$

Conversion between Laplace and z-Transforms

Method 3

- one of following approximation rules can be used:

$$\textbf{Euler forward}: \; s \approx \frac{z-1}{T} \qquad \textbf{Euler backward}: \; s \approx \frac{z-1}{z\,T} \qquad \textbf{Tustin}: \; s \approx \frac{2}{T}\frac{z-1}{z+1}$$

- Forward (explicit) Euler approach is numerically not efficient (very small T required).
- Especially the **Tustin** transformation is often used in practice.
- However, even this approach has its limitations and the discrete-time closed-loop system performance is only comparable to the continuous-time performance if the sampling intervals are sufficiently small.
- More precisely, as long as the cross-over frequency ω_c and the sampling time T satisfy the inequality

$$T < \frac{\pi}{5\omega_c}$$

MATLAB `c2d` command can be used to convert a continuous system into discrete.

Example

write a MATLAB commands to convert $G(s) = \dfrac{1}{s^2 + 5s + 6}$ into discrete with a sample period $T = 1$.

```
>> G = tf([1],[1 5 6]);     % continuous time transfer function
>> T = 1;
>> Gd = c2d(G,T,'impulse') % discrete time transfer function

Gd =
0.08555 z - 8.162e-19
-------------------------
z^2 - 0.1851 z + 0.006738

Sample time: 1 seconds
Discrete-time transfer function.
```

Inverse z-Transform

- Given the z-transform, $Y(z)$, of a function, it is required to find the time-domain function $y(n)$.
- There are **two** methods: power series (long division) and partial fractions.

- **power series**: long division.
  - This method involves dividing the denominator of $Y(z)$ into the numerator to obtain a a power series of the form:

$$Y(z) = y_0 + y_1 z^{-1} + y_2 z^{-2} + y_3 z^{-3} + \cdots$$

  - values of $y(n)$ are, directly, the coefficients in the power series.
- **partial fractions**:
  - a partial fraction expansion of $Y(z)$ is found, and then tables of z-transform can be used to determine the inverse z-transform.

Example

use power series method to find the inverse z-transform for:

$$Y(z) = \frac{z^2 + z}{z^2 - 3z + 4}$$

- Dividing the denominator into the numerator gives: $\Rightarrow$
- from coefficients of power series:

$$y_k = \{1, 4, 8, 8, \cdots\}$$

- The required sequence:

$$y(t) = \delta(t) + 4\,\delta(t - T) + 8\,\delta(t - 2T) + 8\,\delta(t - 3T) + \cdots$$

$$
\begin{array}{r|l}
 & 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} \\
z^2 - 3z + 4 & z^2 + z \\
 & z^2 - 3z + 4 \\
 & 4z - 4 \\
 & 4z - 12 + 16z^{-1} \\
 & 8 - 16z^{-1} \\
 & 8 - 24z^{-1} + 32z^{-2} \\
 & 8z^{-1} - 32z^{-2} \\
 & 8z^{-1} - 24z^{-2} + 32z^{-3} \\
 & \ldots
\end{array}
$$

- in MATLAB, you can use the following commands:

```
  Delta = [1 zeros(1 , 4)];
  num = [0  1 1];
  den = [1  -3  4];
  yk = filter(num, den, Delta)

>> yk =
  0     1     4     8     8
```

- **disadvantage** of power series method: it does not give a **closed form** of the resulting sequence.

Inverse z-Transform

Method 2: Partial Fractions

- Looking at z-transform table, $\Rightarrow$
- there is usually a z term in numerator.
- It is therefore more convenient to find the partial fractions of $Y(z)/z$
- then multiply the partial fractions by z to obtain a z term in the numerator.

No.	Continuous Time	Laplace Transform	Discrete Time	z-Transform
1	$\delta(t)$	1	$\delta(k)$	1
2	$1(t)$	$\frac{1}{s}$	$1(k)$	$\frac{z}{z-1}$
3	t	$\frac{1}{s^2}$	kT	$\frac{zT}{(z-1)^2}$ sampling t gives kT, $z\{kT\} = T\,z\{k\}$
4	t^2	$\frac{2!}{s^3}$	$(kT)^2$	$\frac{z(z+1)T^2}{(z-1)^3}$
5	t^3	$\frac{3!}{s^4}$	$(kT)^3$	$\frac{z(z^2+4z+1)T^3}{(z-1)^4}$
6	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	a^k	$\frac{z}{z-a}$ by setting $a = e^{-\alpha T}$.
7	$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$	$1-a^k$	$\frac{(1-a)z}{(z-1)(z-a)}$
8	$e^{-\alpha t}-e^{-\beta t}$	$\frac{\beta-\alpha}{(s+\alpha)(s+\beta)}$	a^k-b^k	$\frac{(a-b)z}{(z-a)(z-b)}$
9	$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$	kTa^k	$\frac{azT}{(z-a)^2}$
10	$\sin(\omega_n t)$	$\frac{\omega_n}{s^2+\omega_n^2}$	$\sin(\omega_n kT)$	$\frac{\sin(\omega_n T)z}{z^2-2\cos(\omega_n T)z+1}$
11	$\cos(\omega_n t)$	$\frac{s}{s^2+\omega_n^2}$	$\cos(\omega_n kT)$	$\frac{z[z-\cos(\omega_n T)]}{z^2-2\cos(\omega_n T)z+1}$
12	$e^{-\zeta\omega_n t}\sin(\omega_d t)$	$\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$	$e^{-\zeta\omega_n kT}\sin(\omega_d kT)$	$\frac{e^{-\zeta\omega_n T}\sin(\omega_d T)z}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$
13	$e^{-\zeta\omega_n t}\cos(\omega_d t)$	$\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$	$e^{-\zeta\omega_n kT}\cos(\omega_d kT)$	$\frac{z[z-e^{-\zeta\omega_n T}\cos(\omega_d T)]}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$
14	$\sinh(\beta t)$	$\frac{\beta}{s^2-\beta^2}$	$\sinh(\beta kT)$	$\frac{\sinh(\beta T)z}{z^2-2\cosh(\beta T)z+1}$
15	$\cosh(\beta t)$	s [illegible]	$\cosh(\beta kT)$	$z[z-\cosh(\beta T)]$ [illegible]

Example

Find the inverse z-transform of

$$Y(z) = \frac{z^2 + 3z - 2}{(z+5)(z-0.8)(z-2)^2}$$

- Rewriting the function as:

$$\frac{Y(z)}{z} = \frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}$$

$$= \frac{A}{z} + \frac{B}{z+5} + \frac{C}{z-0.8} + \frac{D}{(z-2)} + \frac{E}{(z-2)^2}$$

Inverse z-Transform

Method 2: Partial Fractions

$$A = z\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=0} = 0.125,$$

$$B = (z+5)\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=-5} = 0.0056,$$

$$C = (z-0.8)\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=0.8} = 0.16,$$

$$E = (z-2)^2\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=2} = 0.48,$$

$$D = \left[\frac{d}{dz}\frac{z^2+3z-2}{z(z+5)(z-0.8)}\right]\bigg|_{z=2}$$

$$= \frac{(2z+3)z(z+5)(z-0.8)-(z^2+3z-2)(3z^2+8.4z-4)}{[z(z+5)(z-0.8)]^2}\bigg|_{z=2} = -0.29$$

- We can now write Y(z) as:

$$Y(z) = 0.125 + \frac{0.0056z}{z+5} + \frac{0.016z}{z-0.8} - \frac{0.29z}{(z-2)} + \frac{0.48z}{(z-2)^2}$$

- The inverse transform is found from the tables as

$$y(n) = 0.125\,\delta(n) + 0.0056\,(-5)^n + 0.016\,(0.8)^n - 0.29\,(2)^n + 0.24\,n\,(2)^n$$

- Note: for last term, we used the multiplication by k property which is equivalent to a z-differentiation.

- in MATLAB, you can find the partial fraction expansion of a ratio of two polynomials $F(z)$ with:

$$F(z) = \frac{2z^3 + z^2}{z^3 + z + 1}$$

```
num = [2 1 0 0];
den = [1 0 1 1];
[r,p,k] = residue(num,den)

r =
0.5354 + 1.0390i
0.5354 - 1.0390i
-0.0708 + 0.0000i

p =
0.3412 + 1.1615i
0.3412 - 1.1615i
-0.6823 + 0.0000i

k =
2
```

- residue returns the complex roots and poles, and a constant term in k,
- representing the partial fraction expansion

$$F(z) = \frac{0.5354 + 1.0390i}{z - (0.3412 + 1.1615j)} + \frac{0.5354 - 1.0390i}{z - (0.3412 - 1.1615j)} + \frac{-0.0708}{z + 0.6823} + 2$$

Administrative Stuff

- **tutorial** feedback !
- Mini–Projects · · ·
  - Collision Avoidance Robot
  - Course examples using MATLAB (2x)
  - control lighting system according to the
  - number of people in the room
  - Remote controlled robot using Ardunio and bluetooth.
  - Digital Speed Control
  - Wireless Controlled Robot

Thanks for your attention.

Questions?

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