CSE421

Digital Control

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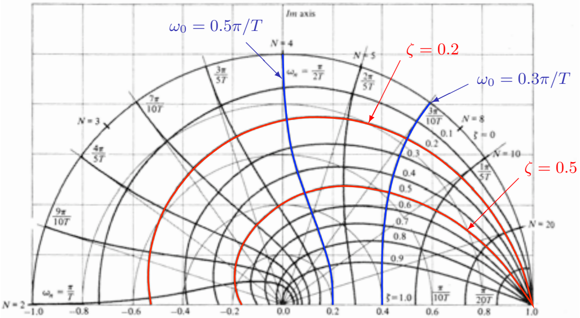
Lecture 3: Linear Discrete Systems Analysis



Zagazig University | Faculty of Engineering | Computer and Systems Dept.

Lecture 3

Linear Discrete Systems Analysis

- Discrete Transfer Functions
- The z-Transform

Discrete Transfer Function

- Compare the continuous system time domain model

$$u(t) = -a_1 \frac{du}{dt} - \cdots - a_n \frac{d^n u}{dt^n} + b_0 e + b_1 \frac{de}{dt} + \cdots + b_m \frac{d^m u}{dt^m} \qquad \text{differential equation}$$

$$\Downarrow \textbf{ Laplace transform } \Downarrow$$

$$D(s) = \frac{U(s)}{D(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{1 + a_1 s + a_2 s^2 + \cdots + a_n s^n}$$

- with the discrete system model:

$$\begin{aligned} u_k &= -a_1 u_{k-1} - \cdots - a_n u_{k-n} + b_0 e_k + \cdots + b_m e_{k-m} \\ &= -\sum_{i=1}^{n} a_i u_{k-i} + \sum_{j=0}^{m} b_j e_{k-j} \qquad \text{recurrence equation} \end{aligned}$$

- **Can we define a transfer function for the discrete system?**

Discrete Transfer Function

Suppose $\mathscr{L}\{u_k = u(kT)\} = U(s) \cdots$ then how can we represent u_{k-1}, u_{k-2}, etc.?

- If $\mathscr{L}\{x(t)\} = X(s)$, then $\mathscr{L}\{x(t-T)\} = e^{-sT}X(s)$, so

$$\begin{aligned} \mathscr{L}\{u_k\} &= U(s) \\ \mathscr{L}\{u_{k-1}\} &= e^{-sT}\,U(s) \\ \mathscr{L}\{u_{k-2}\} &= e^{-2sT}\,U(s) \\ &\vdots \\ \mathscr{L}\{u_{k-n}\} &= e^{-nsT}\,U(s) \end{aligned}$$

- Define the **discrete frequency domain** operator

$$z = e^{sT}$$

$$\begin{aligned} \mathscr{L}\{u_k\} &= U(z) \\ \mathscr{L}\{u_{k-1}\} &= z^{-1}\,U(z) \\ \mathscr{L}\{u_{k-2}\} &= z^{-2}\,U(z) \\ &\vdots \\ \mathscr{L}\{u_{k-n}\} &= z^{-n}\,U(z) \end{aligned}$$

Discrete Transfer Function

Comparison

- system representations:

Continuous

$$u(t) = -a_1 \frac{du}{dt} - \cdots + b_0 e + b_1 \frac{de}{dt} + \cdots$$

Discrete

$$u_k = -a_1 u_{k-1} - \cdots + b_0 e_k + \cdots$$

- operators:

Continuous

$$\frac{du}{dt} \Rightarrow s\, U(s)$$

differential

Discrete

$$u_k \Rightarrow z^{-1}\, U(z)$$

delay

- The operator z is used to denote a **forward shift** by one sampling interval, i.e., $z \cdot x(k) = x(k+1)$.
- The shift operator z is analogous to the Heavyside operator s (differentiation).
- The analogy can be carried further, i.e., the **backward shift operation** is denoted by z^{-1} with $z^{-1} \cdot x(k) = x(k-1)$.

Discrete Transfer Function

- Apply the transformation to the linear recurrence equation:

$$u_k = -a_1 u_{k-1} - a_2 u_{k-2} - \cdots - a_n u_{k-n} + b_0 e_k + b_1 e_{k-1} + \cdots + b_m e_{k-m}$$

$\Downarrow$ **transform** $\Downarrow$

$$U(z) = -a_1 z^{-1} U(z) - a_2 z^{-2} U(z) - \cdots - a_n z^{k-n} U(z) + b_0 E(z) + + b_1 z^{-1} E(z) + \cdots + b_m z^{-m} E(z)$$

- This gives the z–domain transfer function:

$$D(z) = \frac{U(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} = \frac{\prod_{j=1}^{m} (z - z_j)}{\prod_{i=1}^{n} (z - z_i)}$$

zeros : z_j
poles : z_i

  - z_i & z_j are real or in complex conjugate pairs
  - n poles, m zeros, with $(n - m)$ zeros at $z = 0$
  - in general there must be at least as many poles as zeros if it is a **causal** system

Example

Find the discrete transfer function of the PID controller defined in the time and frequency domain by:

$$u(t) = k_p \left[e(t) + \frac{1}{T_I} \int_0^t e(\tau) d\tau + T_D \frac{de(t)}{dt} \right], \qquad U(s) = k_p \left[1 + \frac{1}{T_I\, s} + T_D\, s \right] E(s)$$

- If such a controller is to be realized using a digital computer, the proportional part does not pose any problems.
- The integrator requires more attention. The simplest approach is to approximate it by **Euler forward rule**

$$u(k\,T) = \int_0^{kT} e(\tau) d\tau \approx \sum_{i=0}^{k-1} e(i\,T) \cdot T$$

- For sufficiently small T this approximation yields a controller which produces a closed-loop behavior similar to the one observed in continuous time.

Numerical Integration

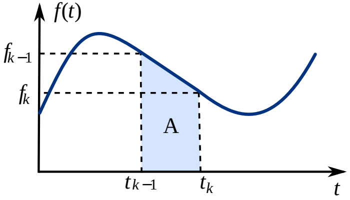
Reminder!

- numerical integration of continuous time $f(t)$ from $t = 0$ to t:

$$y = \int_0^t f(\tau)\, d\tau$$

- We assume that the integral from $t = 0$ to $t = t_{k-1}$ is known,
- using only samples $f_0, f_1, \cdots, f_{k-1}, f_k$, we approx.the integral by computing the area A of the trapezoid:

$$A = \frac{t_k - t_{k-1}}{2}\,(fk + f_{k-1})$$



- For a constant step size T, thus

$$u_k = y = u_{k-1} + \frac{T}{2}\,(fk + f_{k-1})$$

Numerical Differentiation

Reminder! Euler's approximation

$$\frac{dx}{dt} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} \quad \Rightarrow \quad \frac{dx}{dt} \approx \frac{x_{k+1} - x_k}{T}$$

For small enough T, this can be used to approximate a continuous controller using a discrete controller

1. Laplace transform $\longrightarrow$ differential equation

$$\text{e.g.} \quad D(s) = \frac{U(s)}{E(s)} = \frac{k(s+a)}{(s+b)} \quad \Rightarrow \quad \frac{du}{dt} + bu = k(\frac{de}{dt} + ae)$$

2. Differential equation $\longrightarrow$ difference equation

$$\text{e.g.} \quad \frac{u_{k+1} - u_k}{T} + bu_k = k(\frac{e_{k+1} - e_k}{T} + ae_k) \quad \Rightarrow \quad \begin{aligned} u_{k+1} &= (1 - bT)u_k + Ke_{k+1} + K(aT - 1)e_k \\ &= -a_1 u_k + b_0 e_{k+1} + b_1 e_k \end{aligned}$$

Discrete Transfer Function

Example: PI controller

- In a delay–free (the delay can be included in the plant dynamics) system, the PID differential–integral equation can be rewritten as:

$$\frac{du}{dt} = k_p \left[\frac{e}{dt} + \frac{1}{T_I} e + T_D \frac{d^2 e}{dt^2} \right]$$

- using Euler's approximation (for 1^{st} and 2^{nd} derivatives), which gives approximate discrete time controller:

$$\frac{u_k - u_{k-1}}{T} = k_p \left[\frac{e_k - e_{k-1}}{T} + \frac{1}{T_I} e_k + T_D \frac{e_k - 2e_{k-1} + e_{k-2}}{T^2} \right]$$

$$u_k = u_{k-1} + k_p \left[\left(1 + \frac{T}{T_I} + \frac{T_D}{T}\right) e_k - \left(1 + \frac{2T_D}{T}\right) e_{k-1} + \frac{T_D}{T} e_{k-2} \right]$$

- which can be written in a linear recurrence form as:

$$u_k = -a_1 u_{k-1} + b_0 e_k + b_1 e_{k-1} + b_2 e_{k-2}$$

- For sufficiently small T, this approximation yields a controller which produces a closed-loop behavior similar to the one observed in continuous time.

The z-transform

- So far we have considered z^1 as a **delay operator** acting on sequences
- to find $E(z)$ from $e(kT)$ we need to define the z-transform of the sequence:

$$\begin{aligned}\mathscr{Z}\{e(kT)\} &= \mathscr{Z}\{e_k\} = E(z) \\ &= \sum_{k=0}^{\infty} e(kT)z^{-k} = \sum_{k=0}^{\infty} e_k z^{-k} = e_0 + e_1 z^{-1} + e_2 z^{-2} + e_3 z^{-3} + \cdots\end{aligned}$$

- the **coefficients** of this power series are the **samples** e_k at different sampling instants.
- This is a single-sided z-transform (i.e. all variables are assumed to be **zero** for $k < 0$).

z-Transforms of Standard Discrete-Time Signals

- we now obtain the z-transforms of commonly used discrete-time signals (sampled step, exponential, and the discrete time impulse).
- The following identities are used repeatedly to derive several important results:

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a},\ a > 1, \qquad \sum_{k=0}^{n} a^k = \frac{1}{1 - a}\ ,\ 0 < a < 1$$

Example: discrete-time impulse

derive the z-transform of the signal $u(k) = \delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$

- Applying the definition of the z-transform, we get:

$$\mathscr{Z}\{u(k)\} = \sum_{k=0}^{\infty} u_k\, z^{-k} = 1 + 0 + 0 + \cdots = 1$$

z-Transforms of Standard Discrete-Time Signals

Example: decaying exponential

derive the z-transform of sampling the signal $x(t) = C\,e^{-at}\,\mathrm{u}(t)$, $\mathrm{u}(t) =$ unit step at $t = 0$

- sampling the signal $x(t)$:

$$x^*(t) = x(kT) = x_k = C\,e^{-akT}, k \in \mathbb{Z}^*$$

- take the z-transform:

$$\mathscr{Z}\{x(kT)\} = X(z) = C\sum_{k=0}^{\infty} e^{-akT}\,z^{-k} = C\sum_{k=0}^{\infty}\left(e^{-aT}\,z^{-1}\right)^k$$

- this is a **geometric series**[1] which converges if $|z| > e^{-aT}$:

$$X(z) = \frac{C}{1 - e^{-aT}\,z^{-1}} = \frac{Cz}{z - e^{-aT}}$$

- z-transform of exponential = rational polynomial (like Laplace)

[1] $U(z) = 1 + a\,z^{-1} + a^2\,z^{-2} + \cdots$, for $0 < a < 1$ then $U(z) = \frac{1}{1-(a/z)}$

z-Transforms of Standard Discrete-Time Signals

Example: trapezoidal integration

apply the z-transform to the difference equation of trapezoidal integration: $u_k = u_{k-1} + \frac{T}{2}(e_k + e_{k-1})$

- We can do this by multiplying the difference equation by z^{-k} and summing from 0 to ∞

$$\underbrace{\sum_{k=0}^{\infty} u_k\, z^{-k}}_{U(z)} = \underbrace{\sum_{k=0}^{\infty} u_{k-1}\, z^{-k}}_{\Downarrow} + \frac{T}{2}(\underbrace{\sum_{k=0}^{\infty} e_k z^{-k}}_{E(z)} + \sum_{k=0}^{\infty} e_{k-1} z^{-k})$$

$$\sum_{k=0}^{\infty} u_{k-1} z^{-k} = \sum_{l=-1}^{\infty} u_l z^{-l-1} = z^{-1} \underbrace{\sum_{l=-1}^{\infty} u_l z^{-l}}_{u_{-1}=0} = z^{-1} \sum_{l=0}^{\infty} u_l z^{-l} = z^{-1} U(z)$$

$$U(z) = z^{-1} U(z) + \frac{T}{2}\left[E(z) + z^{-1} E(z)\right]$$

$$\frac{U(z)}{E(z)} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{T}{2}\frac{z+1}{z-1} \quad \Rightarrow \quad \text{Transfer function}$$

Tables of Laplace and z-Transforms, and z-Transform Properties

No.	Continuous Time	Laplace Transform	Discrete Time	z-Transform	
1	$\delta(t)$	1	$\delta(k)$	1	
2	$1(t)$	$\frac{1}{s}$	$1(k)$	$\frac{z}{z-1}$	
3	t	$\frac{1}{s^2}$	kT	$\frac{zT}{(z-1)^2}$	sampling t gives kT, z{kT} = T z{k}
4	t^2	$\frac{2!}{s^3}$	$(kT)^2$	$\frac{z(z+1)T^2}{(z-1)^3}$	
5	t^3	$\frac{3!}{s^4}$	$(kT)^3$	$\frac{z(z^2+4z+1)T^3}{(z-1)^4}$	
6	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	a^k	$\frac{z}{z-a}$	by setting $a = e^{-\alpha T}$.
7	$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$	$1-a^k$	$\frac{(1-a)z}{(z-1)(z-a)}$	
8	$e^{-\alpha t}-e^{-\beta t}$	$\frac{\beta-\alpha}{(s+\alpha)(s+\beta)}$	a^k-b^k	$\frac{(a-b)z}{(z-a)(z-b)}$	
9	$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$	kTa^k	$\frac{az\,T}{(z-a)^2}$	
10	$\sin(\omega_n t)$	$\frac{\omega_n}{s^2+\omega_n^2}$	$\sin(\omega_n kT)$	$\frac{\sin(\omega_n T)z}{z^2-2\cos(\omega_n T)z+1}$	
11	$\cos(\omega_n t)$	$\frac{s}{s^2+\omega_n^2}$	$\cos(\omega_n kT)$	$\frac{z[z-\cos(\omega_n T)]}{z^2-2\cos(\omega_n T)z+1}$	
12	$e^{-\zeta\omega_n t}\sin(\omega_d t)$	$\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$	$e^{-\zeta\omega_n kT}\sin(\omega_d kT)$	$\frac{e^{-\zeta\omega_n T}\sin(\omega_d T)z}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$	
13	$e^{-\zeta\omega_n t}\cos(\omega_d t)$	$\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$	$e^{-\zeta\omega_n kT}\cos(\omega_d kT)$	$\frac{z[z-e^{-\zeta\omega_n T}\cos(\omega_d T)]}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$	
14	$\sinh(\beta t)$	$\frac{\beta}{s^2-\beta^2}$	$\sinh(\beta kT)$	$\frac{\sinh(\beta T)z}{z^2-2\cosh(\beta T)z+1}$	
15	$\cosh(\beta t)$	$\frac{s}{s^2-\beta^2}$	$\cosh(\beta kT)$	$\frac{z[z-\cosh(\beta T)]}{z^2-2\cosh(\beta T)z+1}$	

No.	Property	Formula
1	Linearity	$\mathscr{Z}\{\alpha f_1(k)+\beta f_2(k)\} = \alpha F_1(z)+\beta F_2(z)$
2	Time Delay	$\mathscr{Z}\{f(k-n)\} = z^{-n}F(z)$
3	Time Advance	$\mathscr{Z}\{f(k+1)\} = zF(z)-zf(0)$ $\mathscr{Z}\{f(k+n)\} = z^nF(z)-z^nf(0)-z^{n-1}f(1)\cdots-zf(n-1)$
4	Discrete-Time Convolution	$\mathscr{Z}\{f_1(k)*f_2(k)\} = \mathscr{Z}\left\{\sum_{i=0}^{k} f_1(i)f_2(k-i)\right\} = F_1(z)F_2(z)$
5	Multiplication by Exponential	$\mathscr{Z}\{a^{-k}f(k)\} = F(az)$
6	Complex Differentiation	$\mathscr{Z}\{k^m f(k)\} = \left(-z\frac{d}{dz}\right)^m F(z)$
7	Final Value Theorem	$f(\infty) = \mathscr{Lim}_{k\to\infty} f(k) = \mathscr{Lim}_{z\to 1}(1-z^{-1})F(z) = \mathscr{Lim}_{z\to 1}(z-1)F(z)$
8	Initial Value Theorem	$f(0) = \mathscr{Lim}_{k\to 0} f(k) = \mathscr{Lim}_{z\to\infty} F(z)$

Administrative Stuff

- More on z-transforms will be given in **tutorials**.
- Mini–Projects $\cdots$

Thanks for your attention.

Questions?

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