CSE421

Digital Control

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Lecture 2: **Linear Discrete Systems Analysis**

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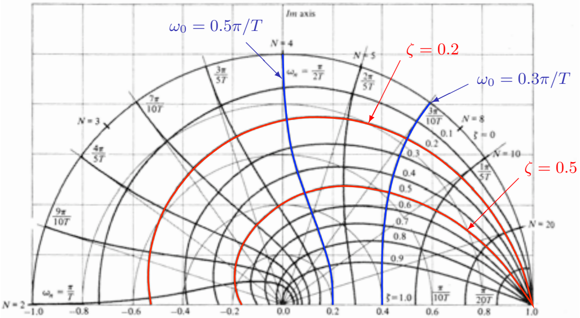

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Lecture 2

Linear Discrete Systems Analysis

- Difference Equations
- Solving Difference Equations

Difference Equations

- The primary new component of discrete or digital systems is the notion of **time discretization**.
  - we are not dealing with variables which are functions of time, now we have **sequences of discrete numbers**.
  - These **discrete numbers** may come from **sampling** a continuous variable, or they may be generated within a **computer**.
- the tools that were used in the analysis of continuous variables will no longer work. We **need new methods**.

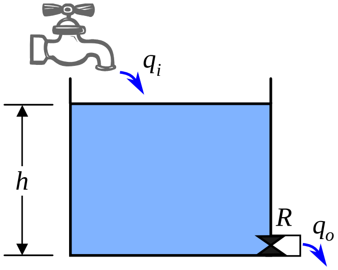
Difference Equations

Analog systems with piecewise constant inputs

Example

Consider the tank control system. It is necessary to maintain a constant fluid level h by adjusting the fluid flow rate into the tank around q_i.

- Obtain an analog mathematical model of the tank, and
- obtain a discrete-time model for the system with piecewise constant inflow perturbation q_i and output h.



outlet volumetric flow through the valve:

$$q_o = c_d a \sqrt{2gh}$$

c_d: valve discharge coefficient

a: valve cross section area

Difference Equations

Analog systems with piecewise constant inputs

- rate of change in height is proportional to net flow:

$$A\frac{dh}{dt} = q_i - q_o \quad \text{sub. for } q_o: \quad A\frac{dh}{dt} = q_i - R\sqrt{h}$$

- for small changes in h this equation can be linearized:

$$\frac{dh}{dt} + a\,h = \frac{1}{A}\,q_i$$

- This 1^{st} order ODE can be solved as:

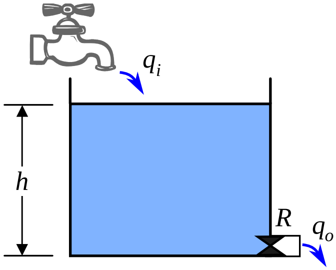
$$e^{-(t-t_0)/a}\,h(t_0) + \frac{1}{A}\int_{t_0}^{t} e^{-(\tau-t_0)/a} q_i(\tau)d\tau$$

- for piecewise constant inflow q_i:

$$q_i(t) = \text{constant} = q_i(k) \quad : t \in [kT,\,(k+1)T]$$

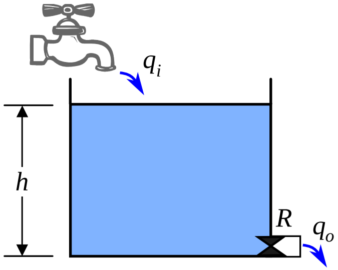
A: tank cross section area

$R = c_d a\sqrt{2g}$: valve resistance



Then, we can solve the analog Eq. over any sampling period to obtain:

$$h(k+1) = e^{-T/a}\, h(k) + R\left[1 - e^{-T/a}\right] q_i(k) \quad k = 0, 1, 2, \cdots$$



- The discrete-time model obtained in this Eq. is known as a **difference equation**.
- the Eq. is linear time–invariant (the analog plant is LTI)
- Next, we briefly discuss difference equations, and then we introduce a transform used to solve them.
- physical **continuous systems** are modeled **differential equations**, **discrete systems** are represented by **difference equations**.

Difference Equations

- The nonlinear difference equation with forcing function $u(k)$:

$$
\begin{aligned}
y(k+n) = f\,[&y(k+n-1), y(k+n-2), \cdots, y(k+1), y(k);\\
&u(k+n), u(k+n), u(k+n-1), \cdots, u(k+1), u(k)]
\end{aligned}
$$

- The **linear** form of this equation:

$$
\begin{aligned}
y(k+n) + a_{n-1}y(k+n-1) + a_{n-1}y(k+n-1) + \cdots, a_1 y(k+1) + a_0 y(k)\\
= u(k+n), u(k+n), u(k+n-1) + \cdots + u(k+1), u(k)
\end{aligned}
$$

- **order** of Eq. is n (difference between highest and lowest time arguments of $y(\cdot)$ and $u(\cdot)$)
- If coefficients $a_i, b_i, i = 0, 1, 2, \cdots$, are constant, the Eq. is **linear time invariant** (LTI).
- If the forcing function $u(k)$ is zero, the Eq. is **homogeneous**.
- If the initial conditions and input are known, a difference equation can be simulated by simply evaluating the equation.

Solving Difference Equations

Example

determine the order of the equation. Is the equation (a) linear, (b) time invariant, or (c) homogeneous?
$y(k+4) + \sin(0.4k)y(k+1) + 0.3y(k) = 0$

- 4^{th} order, homogeneous, linear, time–varying.

Example

Consider the system described by the difference equation: $u_k = u_{k-1} + u_{k-2}$
with $u_0 = 1$ and $u_1 = 1$ computes the output sequence u_k

- by **recursive** substitution in the difference equation, values of u_k can be found as:

k	0	1	2	3	4	5	6	7	$\cdots$
u_k	1	1	2	3	5	8	13	21	$\cdots$

- This is Fibonacci[1] Numbers. (unstable system, why?)

[1] Leonardo Fibonacci of Pisa, who introduced Arabic notation to the Latin world about 1200 A.D.

Solving Difference Equations

- **Remember**! to solve a linear differential equation, we assume a solution of the form $u(t) = Ae^{st} \,:\, s \in \mathbb{C}$
- Similarly, assume solution of difference equation of the form $u(k) = Az^k \,:\, z \in \mathbb{C}, k$ sample index.
- Substituting in the previous difference Eq.:

$$
\begin{aligned}
Az^k &= Az^{k-1} + Az^{k-2} \\
1 &= z^{-1} + z^{-2} \\
0 &= z^2 - z - 1 \qquad \Leftarrow \text{Characteristic Equation}
\end{aligned}
$$

- the two roots of the characteristic equation are: $z_1 = -0.6183, z_2 = 1.6183$[2]
- The general solution is therefore: $u(k) = A_1 z_1^k + A_2 z_2^k$
- $A_1 = 0.276$ and $A_2 = 0.724$ (found from initial conditions). Hence the final solution is:

$$
u(k) = 0.276\,(-0.6183)^k + 0.724\,(1.6183)^k
$$

[2]This is the value of the Golden Ratio

Solving Difference Equations

- Behavior of the two solution **modes**: mode associated with -0.618: decay, but mode associated with root 1.618: grow.

Note on Stability

if any root of the characteristic equation of a discrete system $|z| > 1$, i.e. lies **outside the unit circle** of the z-plane, that system will be **unstable**.

- Any difference equation with a given input can be *solved* in this manner,
- we need **a better way** to **solve or predict** the behavior of difference equations and **discrete systems**.

Thanks for your attention.

Questions?

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