CSE302: Automatic Control

Mathematical Models

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The control systems can be represented with a set of mathematical equations known as **mathematical model**. These models are useful for simulation, prediction/forecasting, design/performance evaluation, and analysis and design of control systems. Analysis of control system means finding the output when we know the input and mathematical model. Design of control system means finding the mathematical model when we know the input and the output.

Mathematical Model

A set of mathematical equations (e.g., differential equations) that describes the input-output behavior of a system.

The **lumped parameter** model simplifies the description of the behavior of spatially distributed physical systems into a topology consisting of discrete entities that approximate the behavior of the distributed system under certain assumptions. It is useful in electrical systems (including electronics), mechanical multibody systems, heat transfer, acoustics, control systems, etc.

This simplification reduces the space model of the physical system into ordinary differential equations (ODEs) with a finite number of parameters. There are different types of lumped-parameter models as shown in Figure 1. The following are mostly used:

- Differential equation model (for linear and nonlinear systems)
- Transfer function model (for linear time invariant systems)
- State space model (for linear and nonlinear, SISO, and MIMO systems)

Let us discuss the first two models in this chapter.

1 Differential Equation Model

Differential equation model is a time domain mathematical model of control systems. Follow these steps for differential equation model.

- Apply basic laws to the given control system.
- Get the differential equation in terms of input and output by eliminating the intermediate variable(s).

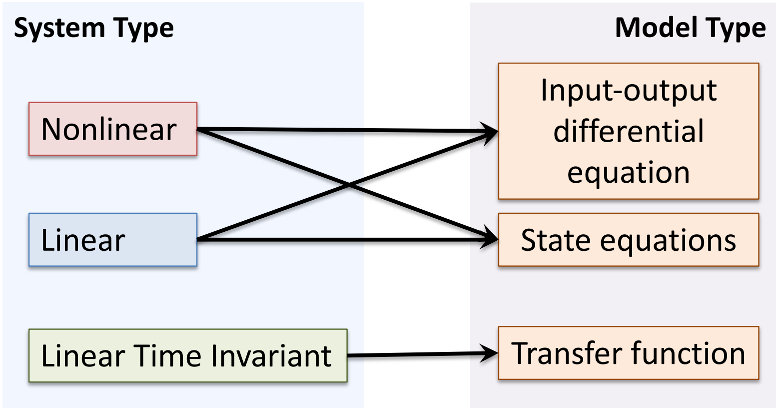

Figure 1: Different types of lumped-parameter models

1.1 Example

Consider the following electrical system as shown in the following figure. This circuit consists of resistor, inductor and capacitor. All these electrical elements are connected in **series**. The input voltage applied to this circuit is v_i and the voltage across the capacitor is the output voltage v_o.

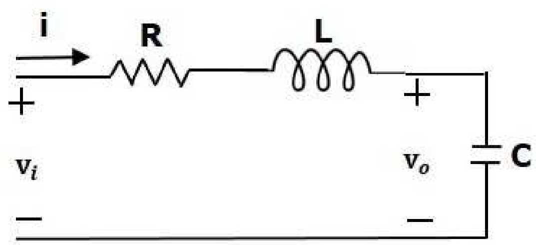

Figure 2: Series

Mesh equation for this circuit is

$$v_i = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + v_o \tag{1}$$

Substitute, the current passing through capacitor $i = c\frac{\mathrm{d}v_o}{\mathrm{d}t}$ in the above equation.

$$v_i = RC\frac{\mathrm{d}v_o}{\mathrm{d}t} + LC\frac{\mathrm{d}^2 v_o}{\mathrm{d}t^2} + v_o \tag{2}$$

$$\frac{\mathrm{d}^2 v_o}{\mathrm{d}t^2} + \left(\frac{R}{L}\right)\frac{\mathrm{d}v_o}{\mathrm{d}t} + \left(\frac{1}{LC}\right)v_o = \left(\frac{1}{LC}\right)v_i \tag{3}$$

The above equation is a second order **differential equation**.

2 Transfer Function Model

Transfer function model is an s-domain mathematical model of control systems. The **Transfer function** of a Linear Time Invariant (LTI) system is defined as the ratio of Laplace transform of output and Laplace transform of input by assuming all the initial conditions are zero.

Transfer Function

the ratio of Laplace transform of output and Laplace transform of input by assuming all the initial conditions are zero.

If $x(t)$ and $y(t)$ are the input and output of an LTI system, then the corresponding Laplace transforms are $X(s)$ and $Y(s)$. Therefore, the transfer function of LTI system is equal to the ratio of $Y(s)$ and $X(s)$.

$$\text{Transfer Function} = \left.\frac{\mathscr{L}\{y(t)\}}{\mathscr{L}\{x(t)\}}\right|_{IC=0} = \frac{Y(s)}{X(s)} \tag{4}$$

The transfer function model of an LTI system is shown in the following Fig. 3. Here, we represented an LTI system with a block having transfer function inside it. And this block has an input $X(s)$ and output $Y(s)$.

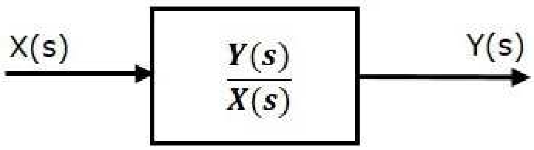

Figure 3: Transfer Function

2.1 Example

Previously, we got the differential equation of an electrical system as:

$$\frac{\mathrm{d}^2 v_o}{\mathrm{d}t^2} + \left(\frac{R}{L}\right)\frac{\mathrm{d}v_o}{\mathrm{d}t} + \left(\frac{1}{LC}\right)v_o = \left(\frac{1}{LC}\right)v_i \tag{5}$$

Apply Laplace transform on both sides:

$$s^2 V_o(s) + \left(\frac{sR}{L}\right)V_o(s) + \left(\frac{1}{LC}\right)V_o(s) = \left(\frac{1}{LC}\right)V_i(s) \tag{6}$$

$$\left[s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}\right]V_o(s) = \left(\frac{1}{LC}\right)V_i(s) \tag{7}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{LC}}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}} \tag{8}$$

Where, $V_i(s)$ and $V_o(s)$ are the Laplace transforms of the input and output voltages v_i and v_o, respectively.

The above equation is a **transfer function** of the **second order** electrical system. The transfer function model of this system is given by Eq. 8. Here, we show a second order electrical system with a block having the transfer function inside it. And this block has an input $V_i(s)$ and an output $V_o(s)$.

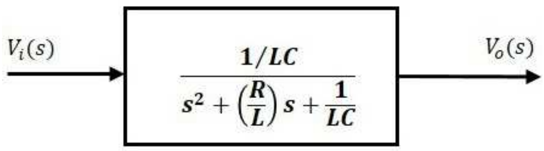

Figure 4: Transfer Function Example

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Block Diagrams

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Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.

3 Basic Elements of Block Diagram

The basic elements of a block diagram are a block, the summing point and the take-off point. Let us consider the block diagram of a closed loop control system as shown in the following figure to identify these elements.

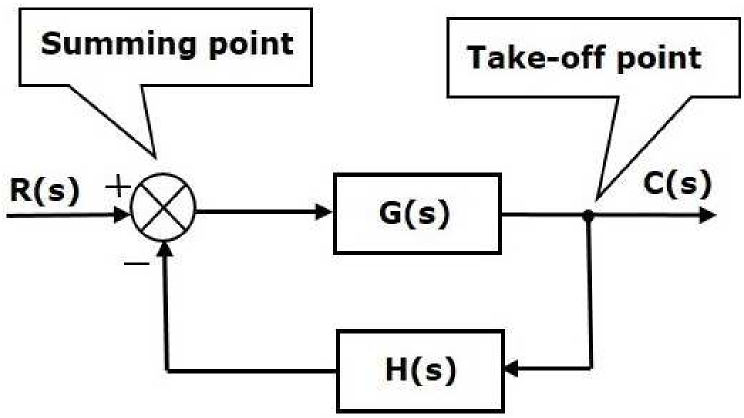

Figure 5: Basic Block Diagram

The above block diagram consists of two blocks having transfer functions G(s) and H(s). It is also having one summing point and one take-off point. Arrows indicate the direction of the flow of signals. Let us now discuss these elements one by one.

3.1 Block

The transfer function of a component is represented by a block. Block has single input and single output. The following figure shows a block having input X(s), output Y(s) and the transfer function G(s). Transfer Function,

$$G(s) = \frac{Y(s)}{X(s)} \quad \Rightarrow Y(s) = G(s)X(s)$$

Output of the block is obtained by multiplying transfer function of the block with input.

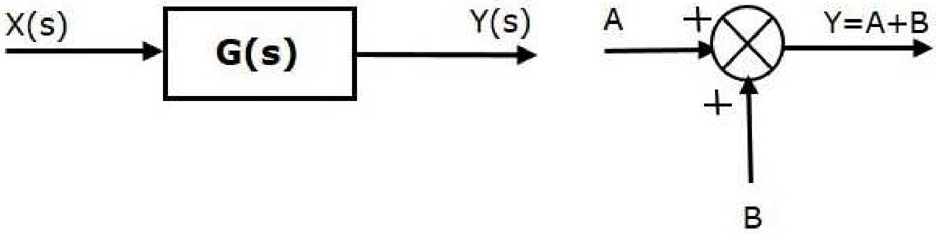

Figure 6: Block (left) and Summing Point (right)

3.2 Summing Point

The summing point is represented with a circle having cross (X) inside it. It has two or more inputs and single output. It produces the **algebraic sum** of the inputs. It also performs the summation or subtraction or combination of summation and subtraction of the inputs based on the polarity of the inputs.

Let us see these three operations one by one. Figure. 7(left) shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B have a positive sign. So, the summing point produces the output, Y as **sum of A and B.** i.e.,Y = A + B.

Figure. 7(right) shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B is having negative sign. So, the summing point produces the output **Y** as the **difference of A and B**.

$$Y = A + (-B) = A - B.$$

The following figure shows the summing point with three inputs (A, B, C) and one output (Y).

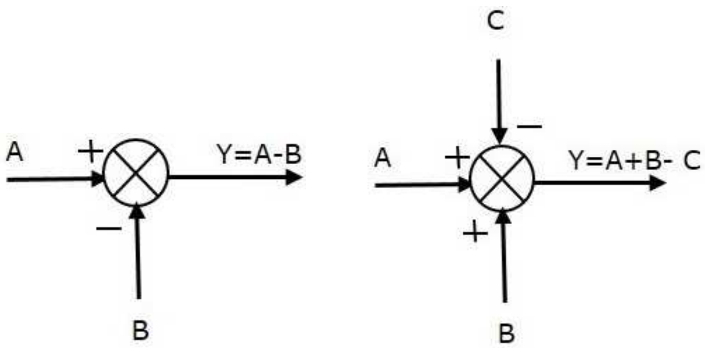

Figure 7: Summing Point examples

Here, the inputs A and B are having positive signs and C is having a negative sign. So, the summing point produces the output **Y** as

$$Y = A + B + (-C) = A + B - C.$$

3.3 Take-off Point

The take-off point is a point from which the same input signal can be passed through more than one branch. That means with the help of take-off point, we can apply the same input to one or more blocks, summing points. In Fig. 8(left), the take-off point is used to connect the same input, R(s) to two more blocks. In Fig. 8(right), the take-off point is used to connect the output C(s), as one of the inputs to the summing point.

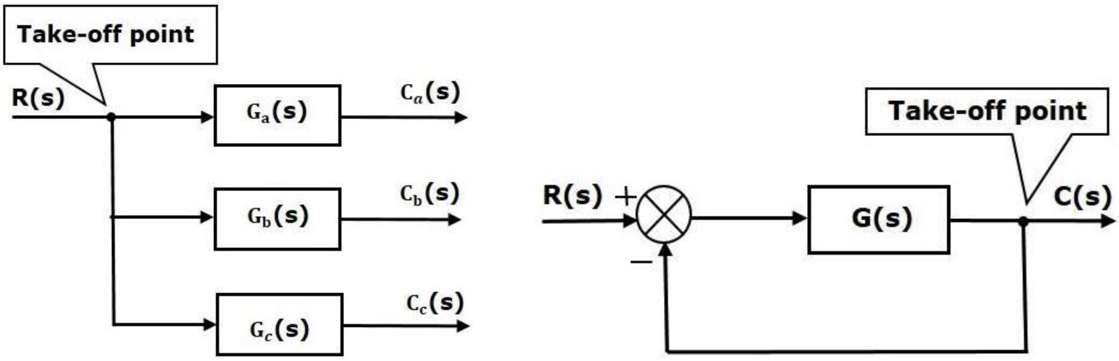

Figure 8: Take-off Point examples

4 Block Diagram Representation of Electrical Systems

In this section, let us represent an electrical system with a block diagram. Electrical systems contain mainly three basic elements — **resistor, inductor and capacitor**. Consider a series of RLC circuit as shown in the following figure. Where, V¡sub¿i¡/sub¿(t) and V¡sub¿o¡/sub¿(t) are the input and output voltages. Let i(t) be the current passing through the circuit. This circuit is in time domain. By applying the Laplace transform to this circuit, will get the circuit in s-domain. The circuit is as shown in the following figure. From the above circuit, we can write

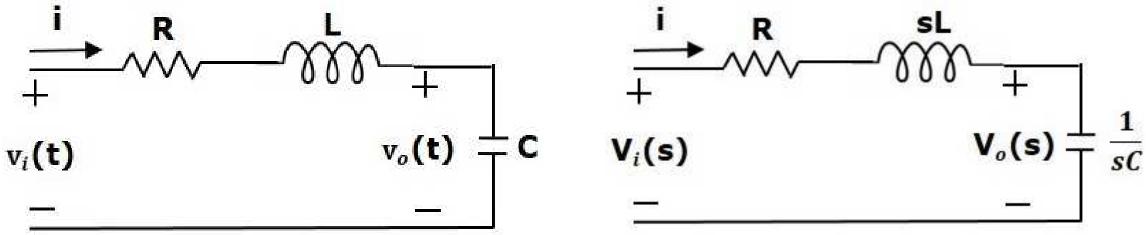

Figure 9: RLC Circuit (left) and its Laplace transform(right)

$$I(s) = \frac{V_i(s) - V_o(s)}{R + sL} \quad \Rightarrow I(s) = \left\{\frac{1}{R + sL}\right\}\{V_i(s) - V_o(s)\} \tag{9}$$

$$V_o(s) = \left(\frac{1}{sC}\right) I(s) \tag{10}$$

Let us now draw the block diagrams for these two equations individually. And then combine those block diagrams properly in order to get the overall block diagram of series of RLC Circuit (s-domain). Equation 1 can be implemented with a block having the transfer function, $\frac{1}{R+sL}$. The input and output of this block are $\{V_i(s) - V_o(s)\}$ and $I(s)$. We require a summing point to get $\{V_i(s) - V_o(s)\}$.

Equation 2 can be implemented with a block having transfer function, $\frac{1}{sC}$. The input and output of this block are $I(s)$ and $V_o(s)$. The block diagram of Eq.9 is shown in Fig.10(lefrt)

The overall block diagram of the series of RLC Circuit (s-domain) is shown in Fig.??.

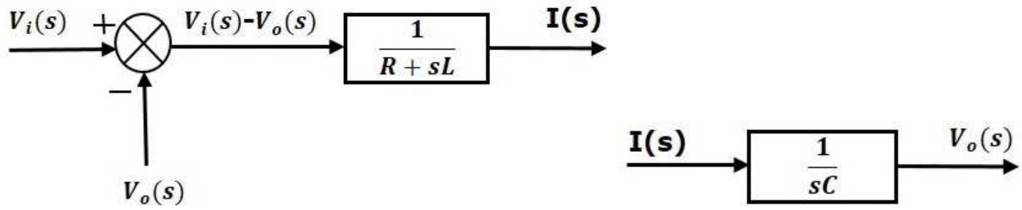

Figure 10: Block diagrams for Eq.9 (left and Eq.10 (right))

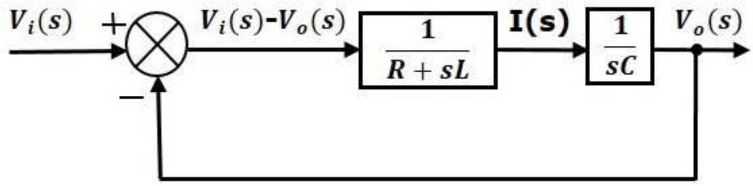

Figure 11: Series RLC Circuit

4.1 Steps for Drawing System Block Diagrams

Similarly, you can draw the **block diagram** of any electrical circuit or system just by following this simple procedure.

- Convert the time domain electrical circuit into an s-domain electrical circuit by applying Laplace transform.
- Write down the equations for the current passing through all series branch elements and voltage across all shunt branches.
- Draw the block diagrams for all the above equations individually.
- Combine all these block diagrams properly in order to get the overall block diagram of the electrical circuit (s-domain).

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Block Diagram Algebra

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Block diagram algebra is nothing but the algebra involved with the basic elements of the block diagram. This algebra deals with the pictorial representation of algebraic equations.

5 Basic Connections for Blocks

There are three basic types of connections between two blocks.

5.1 Series Connection

Series connection is also called **cascade connection**. In the following figure, two blocks having transfer functions $G_1(s)$ and $G_2(s)$ are connected in series.

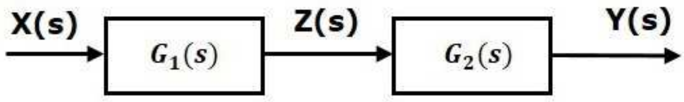

Figure 12: Series Connection

For this combination, we will get the output $Y(s)$ as

$$Y(s) = G_2(s)Z(s)$$

Where, $Z(s) = G_1(s)X(s)$

$$\Rightarrow Y(s) = G_2(s)[G_1(s)X(s)] = G_1(s)G_2(s)X(s)$$

$$\Rightarrow Y(s) = \{G_1(s)G_2(s)\}X(s)$$

Compare this equation with the standard form of the output equation, $Y(s) = G(s)X(s)$. Where, $G(s) = G_1(s)G_2(s)$.

That means we can represent the **series connection** of two blocks with a single block. The transfer function of this single block is the **product of the transfer functions** of those two blocks. The equivalent block diagram is shown below.

Similarly, you can represent series connection of ‘n’ blocks with a single block. The transfer function of this single block is the product of the transfer functions of all those ‘n’ blocks.

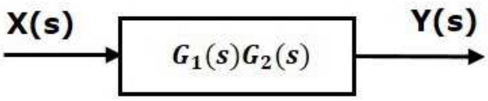

Figure 13: Equivalent Block Diagram

5.2 Parallel Connection

The blocks which are connected in **parallel** will have the **same input**. In the following figure, two blocks having transfer functions $G_1(s)$ and $G_2(s)$ are connected in parallel. The outputs of these two blocks are connected to the summing point.

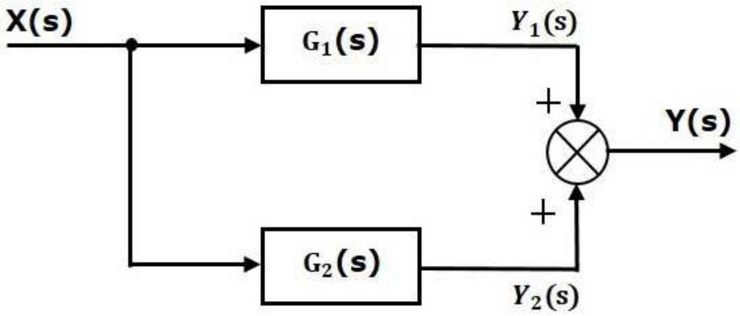

Figure 14: Parallel Connection

For this combination, we will get the output $Y(s)$ as

$$Y(s) = Y_1(s) + Y_2(s)$$

Where, $Y_1(s) = G_1(s)X(s)$ and $Y_2(s) = G_2(s)X(s)$

$$\Rightarrow Y(s) = G_1(s)X(s) + G_2(s)X(s) = \{G_1(s) + G_2(s)\}X(s)$$

Compare this equation with the standard form of the output equation, $Y(s) = G(s)X(s)$.
Where, $G(s) = G_1(s) + G_2(s)$.

That means we can represent the **parallel connection** of two blocks with a single block. The transfer function of this single block is the **sum of the transfer functions** of those two blocks. The equivalent block diagram is shown below.

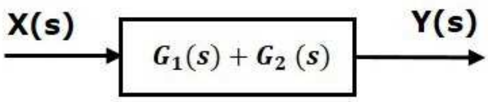

Figure 15: Equivalent Parallel

Similarly, you can represent parallel connection of 'n' blocks with a single block. The transfer function of this single block is the algebraic sum of the transfer functions of all those 'n' blocks.

5.3 Feedback Connection

As we discussed in previous chapters, there are two types of **feedback** — positive feedback and negative feedback. The following figure shows negative feedback control system. Here, two blocks having transfer functions $G(s)$ and $H(s)$ form a closed loop.

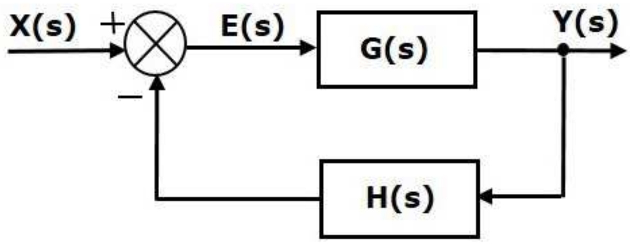

Figure 16: Feedback Connection

The output of the summing point is -

$$E(s) = X(s) - H(s)Y(s)$$

The output $Y(s)$ is:

$$Y(s) = E(s)G(s)$$

Substitute $E(s)$ value in the above equation.

$$Y(s) = \{X(s) - H(s)Y(s)\}G(s)\}$$
$$Y(s)\{1 + G(s)H(s)\} = X(s)G(s)\}$$
$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Therefore, the negative feedback closed loop transfer function is $\frac{G(s)}{1+G(s)H(s)}$

This means we can represent the negative feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the negative feedback. The equivalent block diagram is shown below.

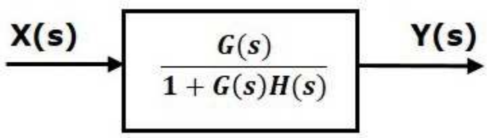

Figure 17: Equivalent Feedback

Similarly, you can represent the positive feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the positive feedback, i.e., $\frac{G(s)}{1-G(s)H(s)}$

6 Block Diagram Algebra for Summing Points

There are two possibilities of shifting summing points with respect to blocks :

- Shifting summing point after the block
- Shifting summing point before the block

Let us now see what kind of arrangements need to be done in the above two cases one by one.

6.1 Shifting Summing Point After the Block

Consider the block diagram shown in the following figure. Here, the summing point is present before the block.

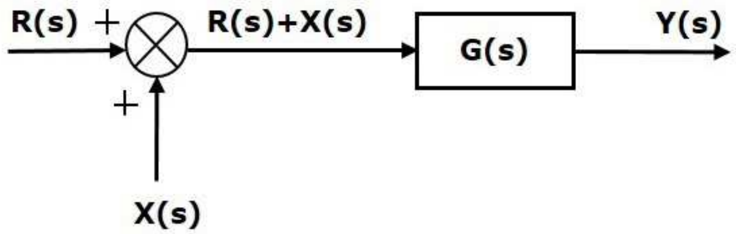

Figure 18: Summing Point Before Block

Summing point has two inputs $R(s)$ and $X(s)$. The output of it is $\{R(s) + X(s)\}$.
So, the input to the block $G(s)$ is $\{R(s) + X(s)\}$ and the output of it is:

$$Y(s) = G(s)\{R(s) + X(s)\} \quad \Rightarrow Y(s) = G(s)R(s) + G(s)X(s) \tag{11}$$

Now, shift the summing point after the block. This block diagram is shown in the following figure.

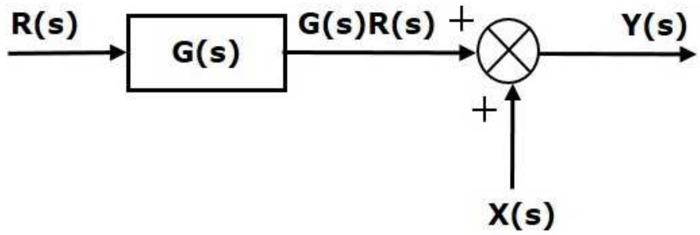

Figure 19: Summing Point After Block

Output of the block $G(s)$ is $G(s)R(s)$.
The output of the summing point is:

$$Y(s) = G(s)R(s) + X(s) \tag{12}$$

Compare Equation 11 and Equation 12.
The first term $G(s)R(s)$ is same in both the equations. But, there is difference in the second term. In order to get the second term also same, we require one more block $G(s)$. It is having the input $X(s)$ and the output of this block is given as input to summing point instead of $X(s)$. This block diagram is shown in the following figure.

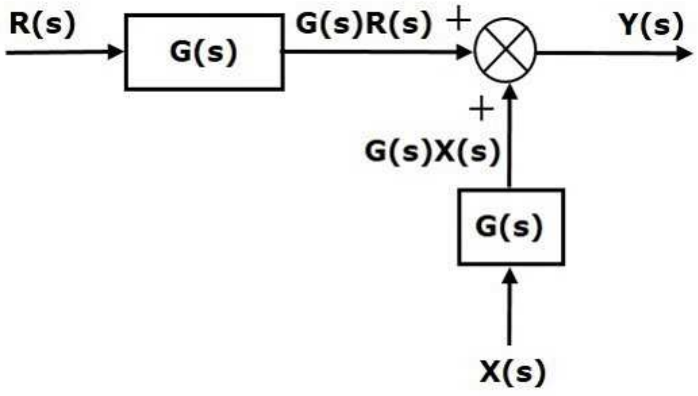

Figure 20: Changed Block

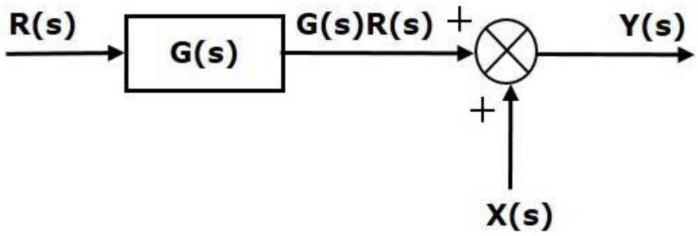

Figure 21: Summing Point After Block

6.2 Shifting Summing Point Before the Block

Consider the block diagram shown in the following figure. Here, the summing point is present after the block.

Output of this block diagram is:

$$Y(s) = G(s)R(s) + X(s) \tag{13}$$

Now, shift the summing point before the block. This block diagram is shown in the following figure.

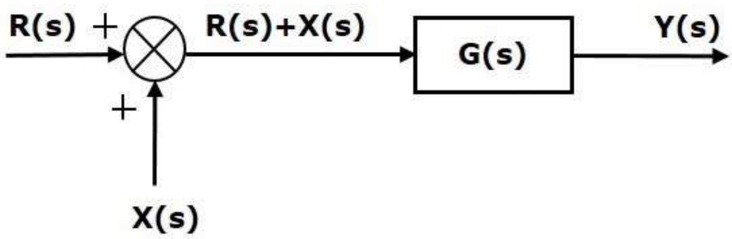

Figure 22: Summing Point Before Block

Output of this block diagram is:

$$Y(S) = G(s)R(s) + G(s)X(s) \tag{14}$$

Compare Equation 13 and Equation 14.

The first term $G(s)R(s)$ is same in both equations. But, there is difference in the second term. In order to get the second term also same, we require one more block $\frac{1}{G(s)}$. It is having the input $X(s)$ and the output of this block is given as input to summing point instead of $X(s)$. This block diagram is shown in the following figure.

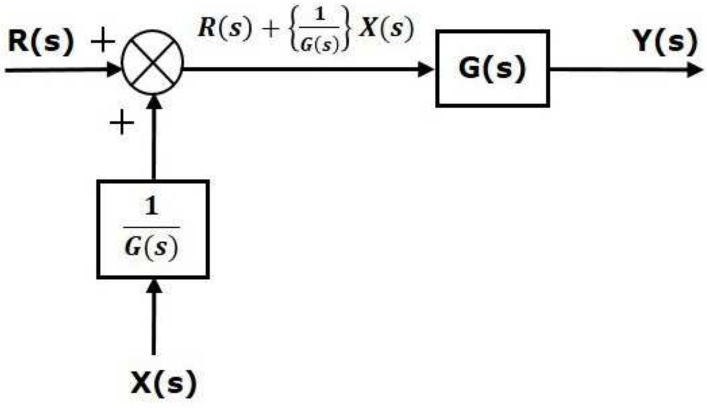

Figure 23: Input Output Block

7 Block Diagram Algebra for Take-off Points

There are two possibilities of shifting the take-off points with respect to blocks :

- Shifting take-off point after the block
- Shifting take-off point before the block

Let us now see what kind of arrangements are to be done in the above two cases, one by one.

7.1 Shifting Take-off Point After the Block

Consider the block diagram shown in the following figure. In this case, the take-off point is present before the block. Here, $X(s) = R(s)$ and $Y(s) = G(s)R(s)$

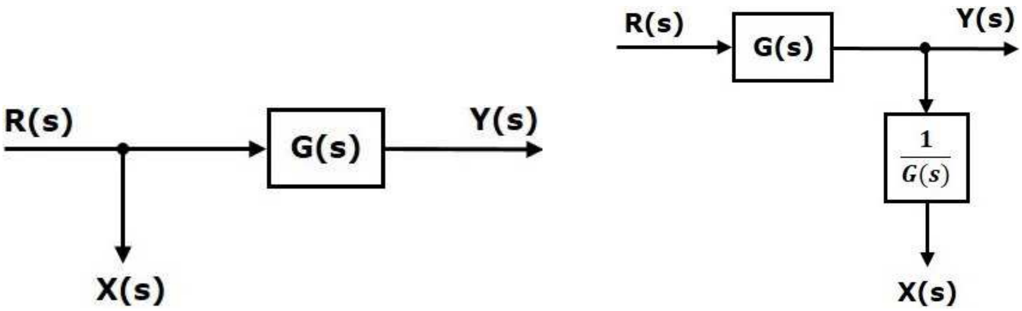

Figure 24: Shifting Take-off After Block

When you shift the take-off point after the block, the output $Y(s)$ will be same. But, there is difference in $X(s)$ value. So, in order to get the same $X(s)$ value, we require one more block $\frac{1}{G(s)}$. It is having the input $Y(s)$ and the output is $X(s)$.

7.2 Shifting Take-off Point Before the Block

Consider the block diagram shown in the following figure. Here, the take-off point is present after the block.

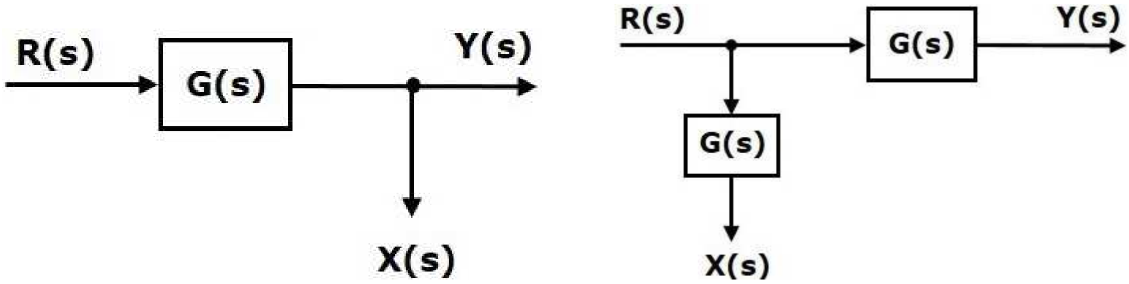

Figure 25: Shifting Take-off Before Block

Here, $X(s) = Y(s) = G(s)R(s)$

When you shift the take-off point before the block, the output $Y(s)$ will be same. But, there is difference in $X(s)$ value. So, in order to get same $X(s)$ value, we require one more block $G(s)$. It is having the input $R(s)$ and the output is $X(s)$.

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Block Diagram Reduction

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The concepts discussed in the previous chapter are helpful for reducing (simplifying) the block diagrams.

8 Block Diagram Reduction Rules

Follow these rules for simplifying (reducing) the block diagram, which is having many blocks, summing points and take-off points.

- **Rule 1** : Check for the blocks connected in series and simplify.
- **Rule 2** : Check for the blocks connected in parallel and simplify.
- **Rule 3** : Check for the blocks connected in feedback loop and simplify.
- **Rule 4** : If there is difficulty with take-off point while simplifying, shift it towards right.
- **Rule 5** : If there is difficulty with summing point while simplifying, shift it towards left.
- **Rule 6** : Repeat the above steps till you get the simplified form, i.e., single block.

Note : The transfer function present in this single block is the transfer function of the overall block diagram.

8.1 Example

Consider the block diagram shown in the following figure. Let us simplify (reduce) this block diagram using the block diagram reduction rules.

Step 1 : Use Rule 1 for blocks G_1 and G_2. Use Rule 2 for blocks G_3 and G_4. The modified block diagram is shown in the following figure.

Step 2 : Use Rule 3 for blocks G_1G_2 and H_1. Use Rule 4 for shifting take-off point after the block G_5. The modified block diagram is shown in the following figure.

Step 3 : Use Rule 1 for blocks $(G_3 + G_4)$ and G_5. The modified block diagram is shown in the following figure.

Step 4 : Use Rule 3 for blocks $(G_3 + G_4)G_5$ and H_3. The modified block diagram is shown in the following figure.

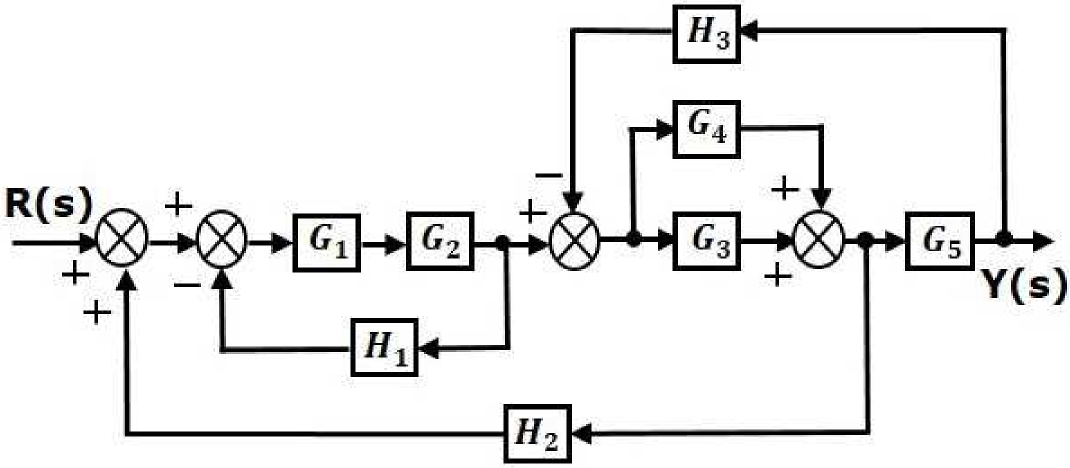

Figure 26: Reduction Diagram

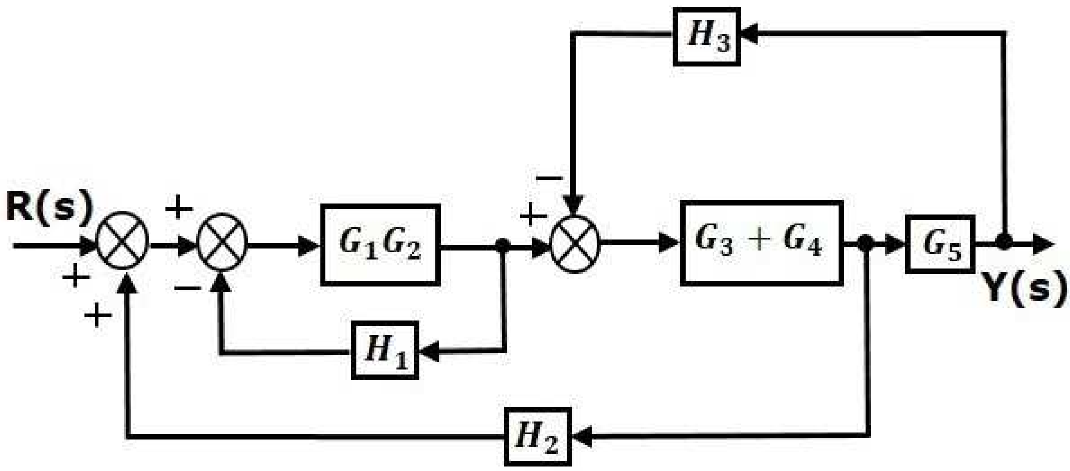

Figure 27: Reduction Step1

Step 5 : Use Rule 1 for blocks connected in series. The modified block diagram is shown in the following figure.

Step 6 : Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.

Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_5^2(G_3+G_4)}{(1+G_1G_2H_1)\{1+(G_3+G_4)G_5H_3\}G_5 - G_1G_2G_5(G_3+G_4)H_2}$$

8.2 Transfer Function of Multi-Input System Block Diagrams

Follow these steps in order to calculate the transfer function of the block diagram having multiple inputs.

- **Step 1** : Find the transfer function of block diagram by considering one input at a time and make the remaining inputs as zero.
- **Step 2** : Repeat step 1 for remaining inputs.

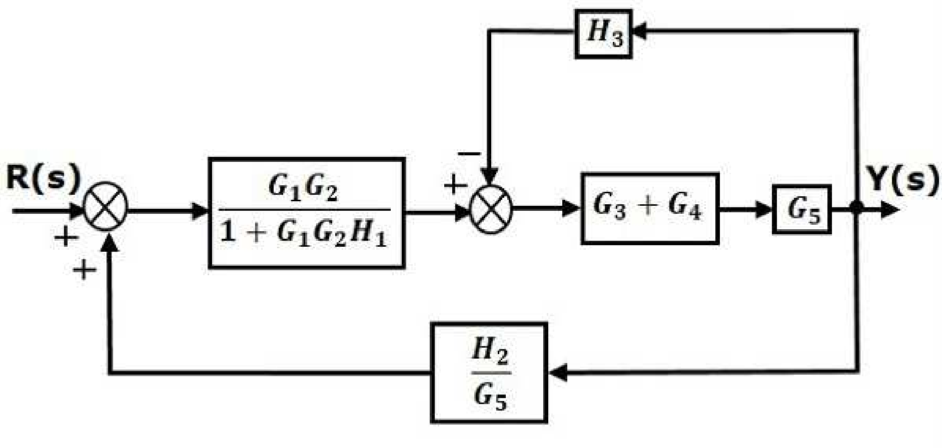

Figure 28: Reduction Step2

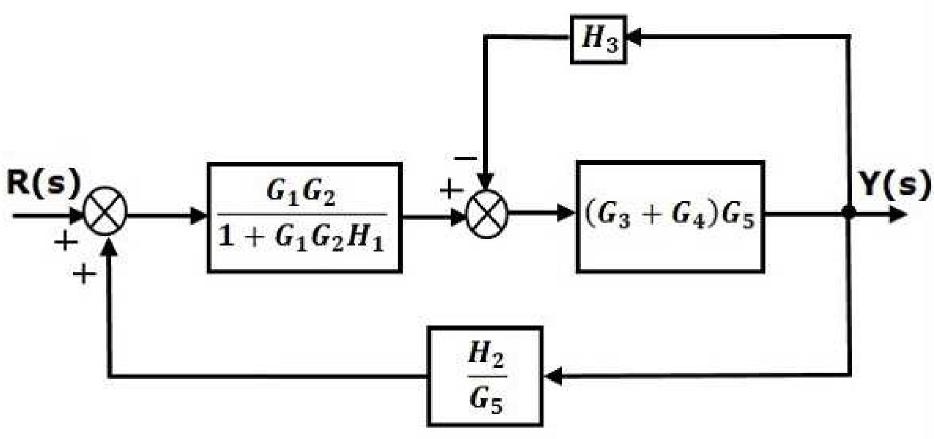

Figure 29: Reduction Step3

- **Step 3** : Get the overall transfer function by adding all those transfer functions.

The block diagram reduction process takes more time for complicated systems. Because, we have to draw the (partially simplified) block diagram after each step. So, to overcome this drawback, use signal flow graphs (representation).

In the next two chapters, we will discuss about the concepts related to signal flow graphs, i.e., how to represent signal flow graph from a given block diagram and calculation of transfer function just by using a gain formula without doing any reduction process.

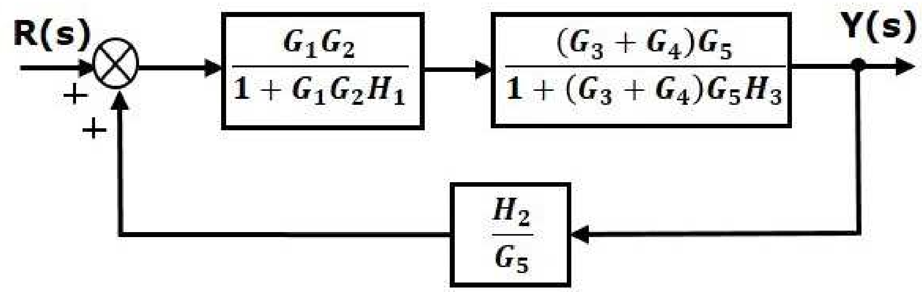

Figure 30: Reduction Step4

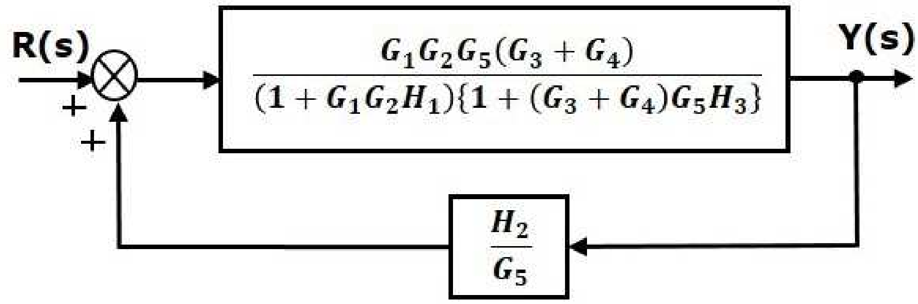

Figure 31: Reduction Step5

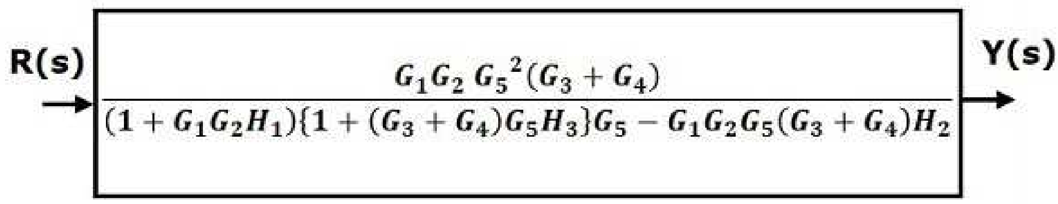

Figure 32: Reduction Step6