CSE302 Automatic Control Engineering

Lecture 5: **Stability Analyses**

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Lecture: 5

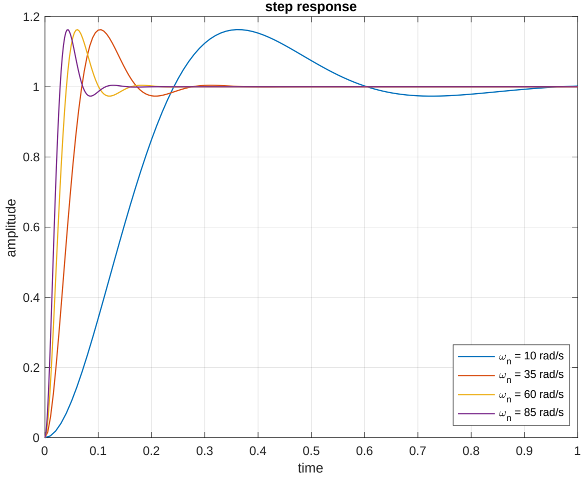
Stability Analyses

- Response of higher-order systems
- Stability of linear systems
- Routh-Hurwitz stability criterion

Time Domain Analysis of second Order Systems

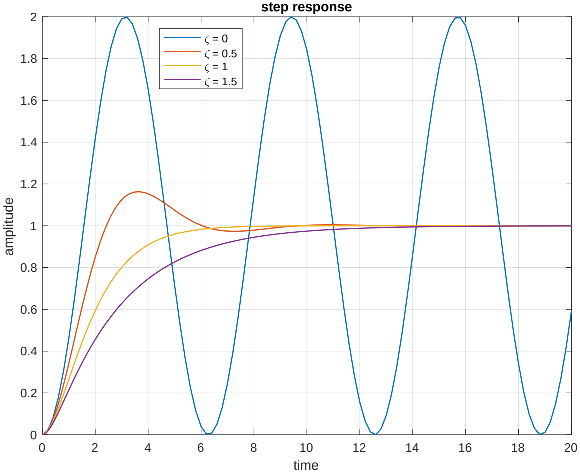
Effect of Natural Frequency

$\zeta = 0.5$ and $\omega_n = 10, 35, 60, 85$ rad/s

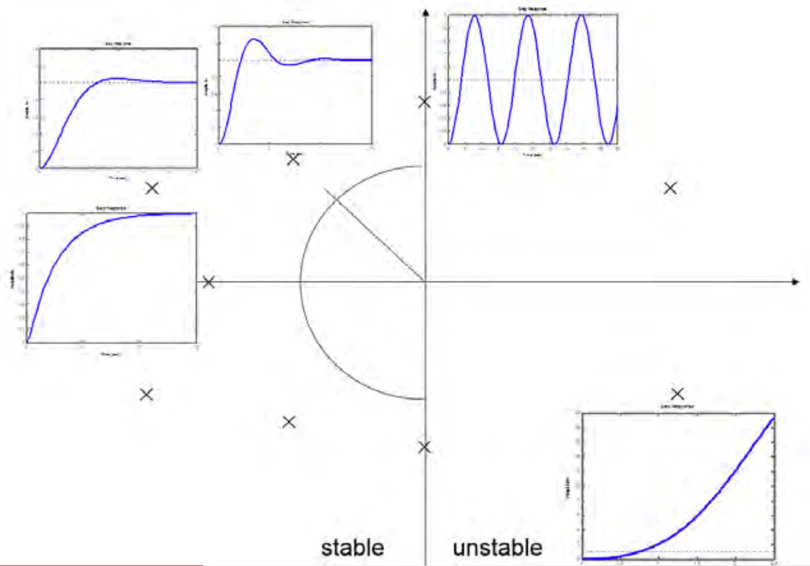


Effect of Damping ratio

$\zeta = 0, 0.5, 1, 1.5$ and $\omega_n = 1$ rad/s



Time Domain Analysis of second Order Systems



Response of higher-order systems

- How can we analyze systems with more zeros, more poles?
- First, write the TF in this standard form:

$$H(s) = K\,\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

- **Location of poles** determines almost everything
- How many cases do we have?

1. For **distinct real** poles:

$$H(s) = \frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \cdots + \frac{\alpha_n}{s-p_n} = \sum_{i=1}^{n} \frac{\alpha_i}{s-p_i}$$

   - it is very easy to derive unit impulse responses as:

$$y(t) = \alpha_1\, e^{-p_1\, t} + \cdots + \alpha_n\, e^{-p_n\, t} = \sum_{i=1}^{n} \alpha_i\, e^{-p_i\, t}$$

   - Transients will vanish **iff** $p_1, \ldots, p_n$ are **negative**

Response of higher-order systems

2. For distinct **real and complex** poles:

$$H(s) = \sum_{i=1}^{q} \frac{\alpha_i}{s - p_i} + \sum_{k=1}^{r} \frac{\beta_k + \gamma_k}{s^2 + 2\sigma_k\, s + \omega_k^2}$$

- Unit-impulse response:

$$y(t) = \sum_{i=1}^{q} \alpha_i\, e^{-p_i\, t} + \sum_{k=1}^{r} c_k\, e^{-\sigma_k t} \sin\left(\omega_k\, t + \theta_k\right)$$

- Similar to the previous case, transients will vanish if all poles are in the left hand side (LHP) of the complex s-plan

Response of higher-order systems

- Each real pole p contributes to an exponential term in any response
- Each complex pair of poles contributes a modulated oscillation
  - The decay of these oscillations depend on whether the real-part of the pole is negative or positive
  - The magnitude of oscillations, contributions depends on residues, hence on zeros

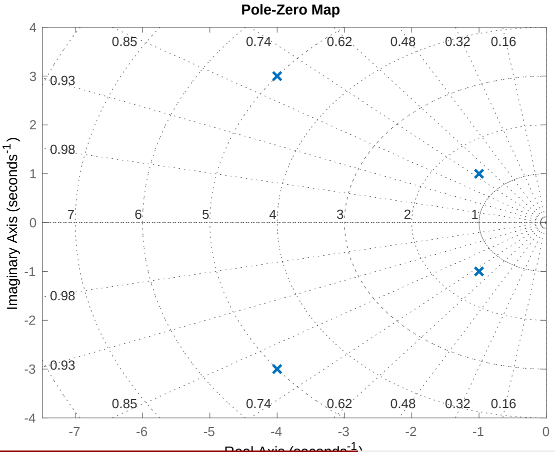
Dominant Poles

the poles **closest to the imaginary axis** are the ones that tend to dominate the response since their contribution takes a longer time to die out.

Dominant Poles

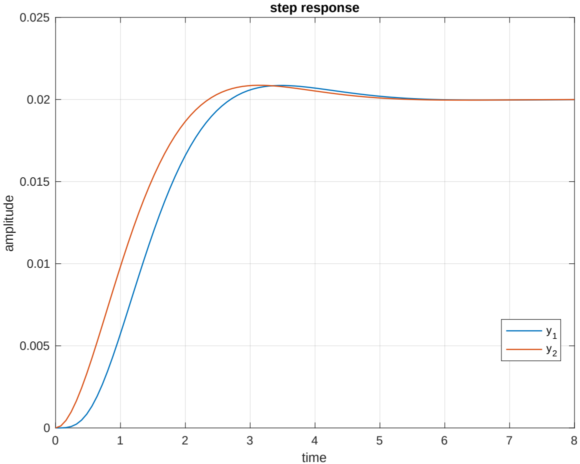
$$H_1 = \frac{1}{(s^2 + 2\,s + 2)\,(s^2 + 8\,s + 25)}$$

$$p_{1,2} = 1 \pm j\,1 \quad p_{3,4} = -4 \pm j\,3$$



$$H_1 = \frac{1}{(s^2 + 2\,s + 2)}$$

$$p_{1,2} = 1 \pm j\,1$$



Stability Analyses

- Stability: one of the most important problems in control
- Suppose that we have the following transfer function of a closed-loop discrete-time system:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{N(s)}{D(s)}$$

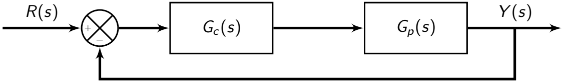
- The system is **stable** if **all** poles* have strictly negative real parts i.e. lie on the **left-hand-side** of the s-plane.

BIBO: Bounded-Input Bounded-Output

System is stable if, under bounded input, its output will converge to a finite value, i.e., transient terms will eventually vanish. Otherwise, it is unstable

*roots of the characteristic equation $D(s) = 0$

Design Problems Related to Stability



- **Stability Criterion:** for a given system (i.e., given $G_c(s)$, $G_p(s)$), determine if it is stable
- **Stabilization**: for a given system that is unstable (i.e., poles of $G_p(s)$ are unstable), design $G_c(s)$ such as $Y(s)/R(s)$ is stable
- Most engineering **design applications** for control systems evolve around this simple, yet occasionally **challenging** idea
- Some systems cannot be stabilized
- For more complex $G_p(s)$, design of $G_c(s)$ is likely to be more complex

Stability Analyses

There are several methods to check the stability of a discrete-time system such as:

- Factorizing $D(s)$ and finding its roots.
- Lyapunov Stability Theorem
- Routh–Hurwitz criterion .

Factorizing Characteristic Equation

- The direct method to check system stability is to factorize the characteristic equation,
  - determine its roots, and check if their **real parts** are all less than 0.
- it is **not usually easy** to factorize the characteristic equation by hand
- we can use MATLAB command `roots`.

$$T_1(s) = \frac{10(s+1)}{s^5 + 3s^3 + 2s + 5} \qquad T_2(s) = \frac{10(s+1)}{s\left(s^4 + s^3 + 3s^2 + s + 2\right)}$$

```
roots([1 0 3 0 2 5])
```

$$\begin{aligned} &-0.9323 \\ &-0.3036 + j\,1.7167 \\ &-0.3036 - j\,1.7167 \\ &0.7697 + j\,1.0827 \\ &0.7697 - j\,1.0827 \end{aligned}$$

Unstable

```
abs(roots([1 1 3 1 2 0]))
```

$$\begin{aligned} &0 \\ &-0.5000 + j\,1.3229 \\ &-0.5000 - j\,1.3229 \\ &-0.0000 + j\,1.0000 \\ &-0.0000 - j\,1.0000 \end{aligned}$$

Stable

Routh–Hurwitz Stability Criterion

- It is a method for determining continuous system stability.
- for any polynomial of any degree, determine number of roots (**not** the exact locations) in the LHS, RHS, or $j\omega$ axis **without having to solve** the polynomial
- Advantages: Less computations

Routh–Hurwitz Criterion

the number of roots of the characteristic equation with positive real parts is equal to the number of **changes in sign** of the first column of the Routh array.

Sufficient and Necessary Condition for Routh-Hurwitz Stability

Sufficient: All the **coefficients** of characteristic array must be **non-zero AND** of the **same sign**
Necessary: all the elements of the **first column** of the Routh array should have the **same sign**.

Routh–Hurwitz Stability Criterion

- Consider the characteristic equation:

$$a_0\, s^n + a_1\, s^{n-1} + a_2\, s^{n-2} + \cdots + a_{n-1}\, s^1 + a_n\, s^0 = 0$$

- if all coefficients of a_i are nonzero and have the same sign, then construct the **Routh-Array**:

$$\begin{array}{c||ccccc}
s^n & \boxed{a_0} & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots & \cdots \\
s^{n-3} & c_1 & c_2 & \vdots & & \\
\vdots & \vdots & \vdots & \vdots & & \\
s^0 & a_n & & & &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}, \qquad b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1} \qquad b_3 = \frac{a_1 a_6 - a_7 a_0}{a_1}, \qquad \cdots$$

$$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}, \qquad c_2 = \frac{b_1 a_5 5 - b_3 a_1}{b_1}, \qquad \cdots$$

- number of roots on the right half-plane is equal to number of sign changes in the first column.

Example

Determine the stability of the systems with the following characteristic polynomials:

a) $s^2 - s + 1$

b) $s^4 + s^3 + s^2 + 1$

c) $-s^4 + s^3 + s^2 + s + 1$

d) $s^4 + 2s^3 + 3s^2 + 4s + 5$

- a. $s^2 - s + 1$ is not stable,
- b. $s^4 + s^3 + s^2 + 1$ is not stable
- c. $-s^4 + s^3 + s^2 + s + 1$ is undetermined

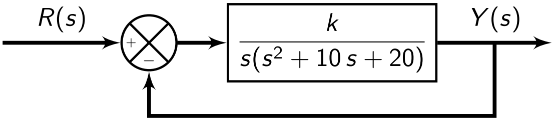
4. for $s^4 + 2s^3 + 3s^2 + 4s + 5$, we construct the Routh array as:

$$
\begin{array}{c||lll}
s^4 & 1 & 3 & 5 \\
s^3 & 2 & 4 & 0 \\
s^2 & 1 = \frac{2\times3-4\times1}{2} & 5 = \frac{2\times5-1\times0}{2} & \\
s^1 & -6 = \frac{1\times4-2\times5}{1} & & \\
s^0 & 5 & &
\end{array}
$$

- number of sign changes $= 2$, then two roots are on the RHS of s-plan $\Rightarrow$ system is **unstable**

Example

Given the next unity-feedback system, find range of k such that the system is stable.



- first, we find the closed loop transfer function

$$T(s) = \frac{k}{s^3 + 10\,s^2 + 20\,s + k} \quad \Rightarrow \quad ch.Eq.: s^3 + 10\,s^2 + 20\,s + k = 0$$

- construct the Routh array as:

$$\begin{array}{c||ll} s^3 & 1 & 20 \\ s^2 & 10 & k \\ s^1 & \frac{-1}{10}(k-200) & \\ s^0 & k & \end{array}$$

- Need no sign change in the first column, so $k < 200$ and $k > 0 \quad \Rightarrow 0 < k < 200$

Routh–Hurwitz Stability Criterion

Special Cases: Case 1: Zero in the first column

- If first element of a row is zero, division by zero would be required to form the next row.
- To avoid this case, zero is replaced with a very small **positive** number (ϵ).

Example: $s^3 + 2s^2 + s + 2 = 0$

$$\begin{array}{c||ll} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx \epsilon & \\ s^0 & 2 & \end{array}$$

- the sign of coefficient above ϵ is the same as the sign under $\Rightarrow$ there are pair of complex roots

Example: $s^3 - 3s + 2 = (s-1)2(s+2) = 0$

$$\begin{array}{c||ll} s^3 & 1 & -3 \\ s^2 & 0 \approx \epsilon & 2 \\ s^1 & \left(-3 - \frac{2}{\epsilon}\right) & \\ s^0 & 2 & \end{array}$$

- the sign of the coefficients above and below ϵ change $\Rightarrow$ system is not stable

- an entire row consists of zeros happens because there is an even polynomial that is a factor of the original polynomial.
- Then we have: two real roots with equal magnitudes and opposite signs and/or two complex conjugate roots

- **Example**: $p(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12 = 0$

s^5	1	11	28	
s^4	5	23	12	
s^3	6.4	25.6		
s^2	3	12		
s^1	0	0		$\Leftarrow$ old row, define **auxiliary polynomial** : $P(s) = 3s^2 + 12$
s^1	6	0		$\Leftarrow$ new row, define auxiliary polynomial : $P'(s) = 6s + 0$
s^0	12			

- roots of auxiliary polynomial: $3s^2 + 12 = 0 \Rightarrow p_{1,2} = \pm j2$ are roots for the original polynomial

Example

Determine if the system with the following characteristic equation stable or not:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

- construct the Routh array as:

$$\begin{array}{c||lll}
s^5 & 1 & 24 & -25 \\
s^4 & 2 & 48 & -50 \\
s^3 & 0 & 0 & \\
s^3 & 8 & 96 & \\
s^2 & 24 & -50 & \\
s^1 & 112.7 & 0 & \\
s^0 & -50 & &
\end{array}$$

s^3 row $0 \quad 0$: $\Leftarrow$ old row, define **auxiliary polynomial** : $P(s) = 2s^4 + 48s^2 - 50$

s^3 row $8 \quad 96$: $\Leftarrow$ new row, define auxiliary polynomial $P'(s) = 8s^3 + 96$

- since there is **one** sign change in the first column, then the system is **not** stable.

Thanks for your attention.

Questions?

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