CSE302 Automatic Control Engineering

Lecture 4: Transient and Steady-State Response Analyses

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Lecture: 4

Transient and Steady-State Response Analyses

- Time Domain Analysis of Control systems
  - General linear systems analysis
  - Responses to different test signals
- First Order Dynamical Systems
- Second Order Dynamical Systems

Order and Type of a system

Order of the system

- Consider a system defined by the transfer function:

$$T(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

- The order of this system is n which is defined by the highest power for s in the denominator.

The system type Number

- It is defined as the number of **poles at the origin** of the **open loop transfer function** $G(s)H(s)$.
- Consider the open loop transfer function of a system as :

$$G(s)H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^c\left(s^n + a_{n-1}s^{n-1} + \cdots + a_0\right)}$$

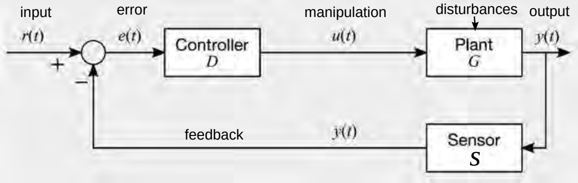
- The system of **type** c and has an **order** of $n + c$.

$$G(s)H(s) = \frac{50}{(s+1)(s+4)} \qquad \Rightarrow \text{System of type 0}$$

$$G(s)H(s) = \frac{10s^2 + 3}{s^2(3s^4 + 2s^3 + s^2 + 4s + 3)} \qquad \Rightarrow \text{System of type 2}$$

Time Domain Analysis

In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.



- It is possible to compute the time response of a system if the nature of **input** and the mathematical **model** of the system are known.
  - Usually, the input signals to control systems are not known fully ahead of time.
  - For example, in a radar tracking system, the position and the speed of the target to be tracked may vary in a random fashion.
- It is therefore difficult to express the actual input signals mathematically by simple equations.

- analyze and characterize input-output behavior
  - Simple idea: want to know how your system is performing?
  - excite it with different test inputs $\Rightarrow$ study the response

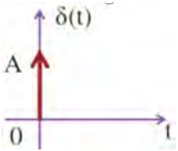
Standard Test Signals

- The characteristics of actual input signals are:
  - a sudden shock, a sudden change, a constant velocity, and constant acceleration.
  - another standard signal of great importance is a sinusoidal signal.
- The dynamic behavior of a system is therefore judged and compared under application of these standard test signals

Standard Test Signals

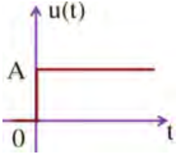
- **Impulse** input: $\delta(t)$, for a sudden shock:

$$\delta(t) = \begin{cases} A, & \text{for } t = 0 \\ 0, & \text{for } t \neq 0 \end{cases} \Rightarrow R(s) = \mathscr{L}\{\delta(t)\} = A$$



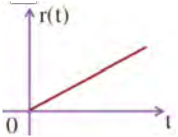
- **Step** input: $u(t)$, characterizes system ability to track sudden input changes:

$$u(t) = \begin{cases} A, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} \Rightarrow U(s) = \mathscr{L}\{u(t)\} = \frac{A}{s}$$



- **Ramp** input: $r(t)$, characterizes system ability to track varying input with aconstant velocity

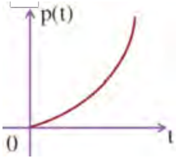
$$r(t) = \begin{cases} A\,t, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} \Rightarrow R(s) = \mathscr{L}\{r(t)\} = \frac{A}{s^2}$$



Standard Test Signals

- **Parabolic** input $p(t)$ imitates the constant acceleration characteristic of actual input signal.

$$p(t) = \begin{cases} \frac{A\,t^2}{2}, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} \Rightarrow P(s) = \mathscr{L}\{p(t)\} = \frac{A}{s^3}$$

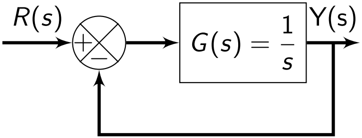


- Why are these important? How is this useful? – Relationship between them:

unit-impulse response	unit-step response	unit-ramp response
$\mathscr{L}\{y(t)\} = T(s)$	$\mathscr{L}\{y(t)\} = \frac{T(s)}{s}$	$\mathscr{L}\{y(t)\} = \frac{T(s)}{s^2}$

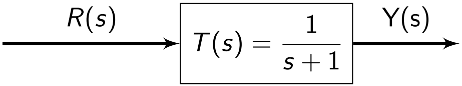
Example

Find the response of the following system to unit impulse, step, and ramp inputs



- First, we find the overall transfer function, $T(s)$
- since $y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\{T(s)R(s)\}$, then:

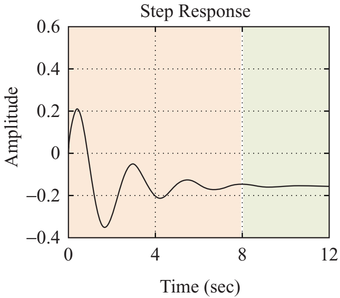
Input $r(t)$	**Response** $y(t)$
$\delta(t)$	e^{-t}
$u(t) = 1^{+}$	$1 - e^{-t}$
$u(t) = t$	$t - 1 + e^{-t}$



Transient and Steady State Responses

- Any output for linear system is decomposed of: $y(t) = y_{ss}(t) + y_{tr}(t)$

- $y_{ss}(t)$: **stead-state** (forced) response — signifies the system's ability to eventually track input signals after few seconds
- $y_{tr}(t)$: **transient** (natural) response — path the output took to reach SS
  - Overly oscillatory $y_{tr}(t)$ is usually bad for systems. Why?
  - Slow transient response is typically undesirable. Why?



Time Domain Analysis of First Order Systems

- First order systems are characterized by:

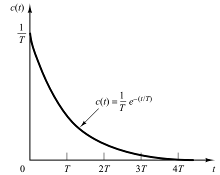
$$T(s) = \frac{Y(s)}{R(s)} = \frac{k}{Ts+1}, \qquad \begin{aligned} T &: \text{time constant}, \\ k &: \text{DC gain} \end{aligned}$$

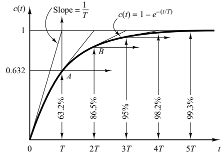
- Impulse response: $y(t) = \mathscr{L}^{-1}\left\{\frac{k/T}{s+1/T}\right\} = \frac{k}{T}e^{\frac{-t}{T}}$

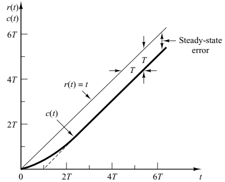
- Step response: $y(t) = \mathscr{L}^{-1}\left\{\frac{k}{s(Ts+1)}\right\} = k\left(1 - e^{\frac{-t}{T}}\right)$

  note: System takes five time constants to reach its final value.

- Ramp response: $y(t) = \mathscr{L}^{-1}\left\{\frac{k}{s^2(Ts+1)}\right\} = k\left(t - T + Te^{\frac{-t}{T}}\right)$





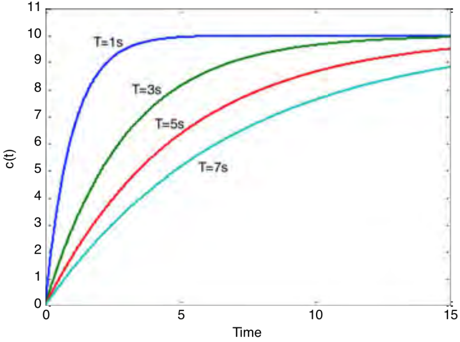


Step response of First Order System:

$$y(t) = \mathscr{L}^{-1}\left\{\frac{k}{s(Ts+1)}\right\} = k\left(1 - e^{\frac{-t}{T}}\right)$$

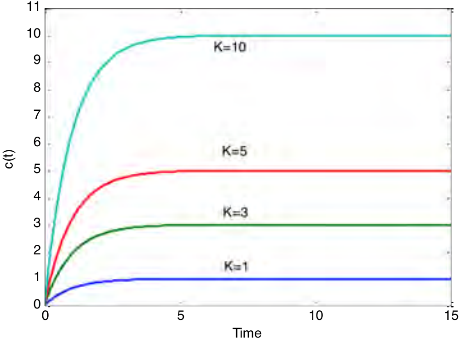
Effect of Time Constant

$k = 10$ and $T = 1, 3, 5, 7$



Effect of DC Gain

$k = 1, 3, 5, 10$ and $T = 1$



Example

Impulse response of a first order system is given by: $y(t) = 3\left(1 - e^{-0.5\,t}\right)$ Find the system time constant, DC gain, transfer function, and step response.

- The Laplace transform of impulse response of a system is its the transfer function.
- Therefore, this system transfer function is:

$$T(s) = \frac{Y(s)}{R(s)} = \mathscr{L}\{3\left(1 - e^{-0.5\,t}\right)\} = \frac{3}{s + 0.5} = \frac{6}{2\,s + 1}$$

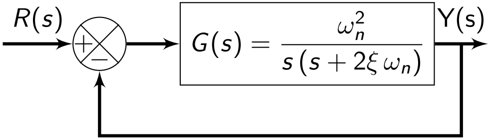
- DC gain $k = 6$, Time constant $T = 2$
- step response:

$$y(t) = \mathscr{L}^{-1}\left\{\frac{6}{2\,s + 1}\frac{1}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{6}{s} - \frac{6}{s + 0.5}\right\} = 6\left(1 - e^{-0.5\,t}\right)$$

Time Domain Analysis of second Order Systems

- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.
- Varying a first-order system parameters (T, K) simply changes the speed and offset of the response
- **Changes in the parameters** of a second-order system can change the **form of the response**.
- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or pure oscillations for its transient response.

Time Domain Analysis of second Order Systems



- Most commonly used time domain performance measures refer to a 2^{nd} order system:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n\, s + \omega_n^2}$$

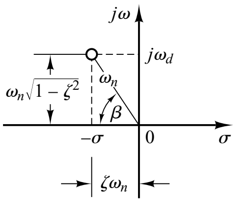
ω_n	undamped natural frequency
ξ	damping ratio

- ξ is a measure of the degree of resistance to change in the system output.
- ω_n is the frequency of oscillation of the system without damping.
- The performance of a control system is usually characterized by its step response.
    - step input is easy to generate and gives the system a nonzero steady-state condition, which can be measured.

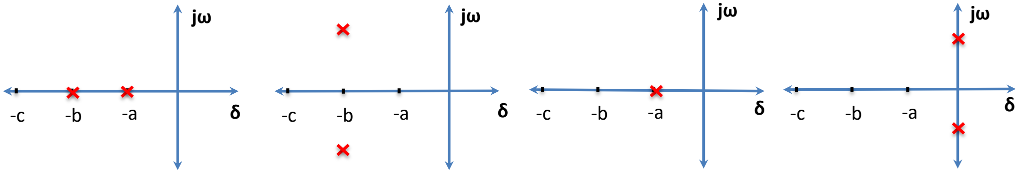
Time Domain Analysis of second Order Systems

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n\, s + \omega_n^2}$$

- System Poles: $s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$



- Depending upon the value of ξ, a second-order system can be set into one of the four categories:
  1. **Overdamped**: the system has **two real distinct** poles ($\xi > 1$).
  2. **Underdamped**: the system has two **complex** conjugate poles ($0 < \xi < 1$)
  3. **Critically damped**: the system has **two real repeated** poles ($\xi = 1$).
  4. **Undamped**: the system has two **pure complex** poles ($\xi = 0$)



Step Response of second Order Systems

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n\, s + \omega_n^2} \qquad \text{for unit-step input } R(s) = \frac{1}{s}$$

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n\, s + \omega_n^2}\, R(s) = \frac{\omega_n^2}{s\,(s^2 + 2\xi\omega_n\, s + \omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n^2\,(1 - \xi^2)} = \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

$$= 1 - \frac{1}{\sqrt{1 - \xi^2}}\, e^{-\xi\omega_n t} \sin\left(\omega_n \sqrt{1 - \xi^2}\; t + \phi\right)$$

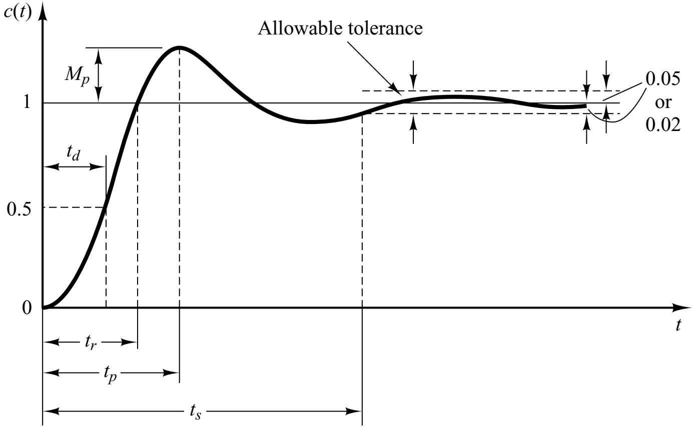
$$\phi = \tan^{-1} \frac{\sqrt{1 - \xi^2}}{\xi} \quad 0 < \xi < 1, \quad 0 < \phi < \frac{\pi}{2},$$

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \text{ damped natural frequency}$$

Unit-step response of overdamped and undamped Systems are left as exercise to you.

Time domain specifications

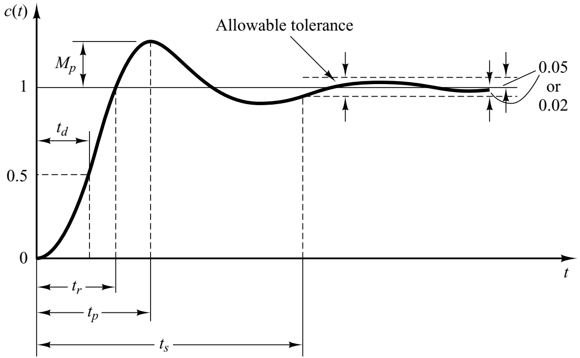
- The typical response of a 2^{nd} order system when excited with a unit step input



- In this response, the performance indices are usually defined: **rise time; peak time; settling time; maximum overshoot; and steady-state error**.

Time domain specifications

Rise time (T_r)

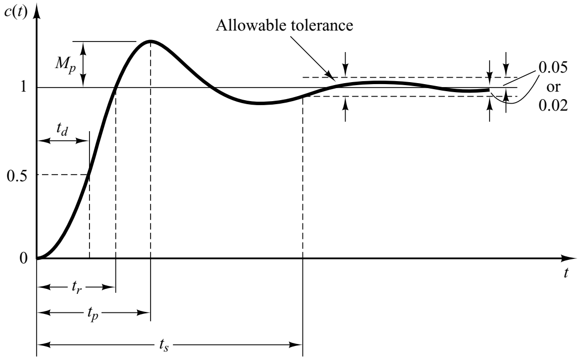


- time required for the response to go from 0 to 100% of its final value.
- a measure of speed of response (smaller rise time ⇒ faster response)

$$T_r = \frac{\pi - \beta}{\omega_d}, \quad \beta = \tan^{-1}\frac{\omega_d}{\xi\omega_n}, \quad \omega_d = \omega_n\sqrt{1-\xi^2}$$

Time domain specifications

Peak time (T_p)

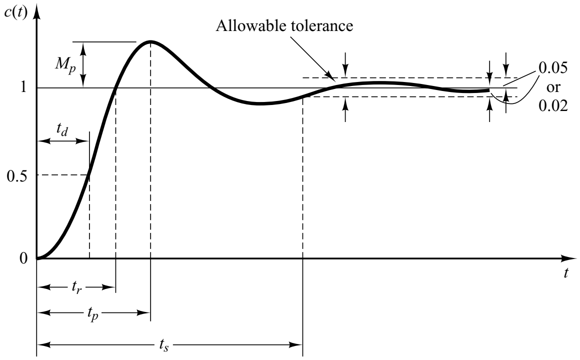


- time required for the response to reach its peak value.
- response is faster when peak time is smaller, but with higher overshoot.

$$T_p = \frac{\pi}{\omega_d}$$

Time domain specifications

Settling time (T_s)

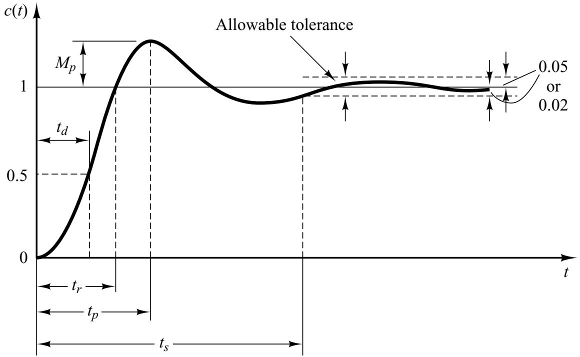


- time required for response to reach and stay within a range (2 or 5%) about final value.

$$T_s = \frac{3}{\xi\omega_n} \quad (\text{for } 5\%) \quad T_s = \frac{4}{\xi\omega_n} \quad (\text{for } 2\%)$$

Time domain specifications

Percent Overshoot (M_p)



- The percent overshoot is defined as:

$$M_p = \frac{y_m - y_{ss}}{y_{ss} - y_0} \times 100\%,$$

y_0 initial, y_m maximum, and y_{ss} is steady state (final) values of the step response, respectively.

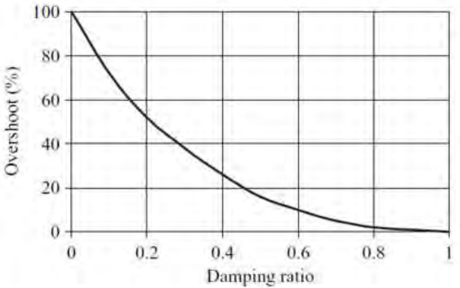
Time domain specifications

Percent Overshoot (M_p)

- For 2^{nd} order system, the percent overshoot is calculated as:

$$M_p = \exp(-\xi\pi/\sqrt{1-\xi^2})$$

- The amount of overshoot depends on the damping ratio (ξ) and directly indicates the relative stability of the system.
  - The lower is the damping ratio, the higher the is maximum overshoot.



The steady-state error (e_{ss})

- difference between the reference input and the output response at steady-state.
- Small e_{ss} is required in most control systems. However, in some systems such as position control, it is important to have zero e_{ss}.
- e_{ss} can be found using the final value theorem:
  If the Laplace transform of the output is $Y(s)$, then final (steady-state) value is given by:

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

- Hence, for a unit step input, e_{ss} is given as:

$$e_{ss} = 1 - \lim_{s \to 0} sY(s)$$

Time domain specifications

Example

Determine the step response **performance indices** of the system with the following closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{1}{s^2 + s + 1}$$

- Comparing the given system with the standard 2^{nd} order transfer function:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- We find that $\omega_n = 1$ rad/s and $\xi = 0.5$. Thus, the damped natural frequency is:

$$\omega_d = \omega_n\sqrt{1 - \xi^2} = 0.866$$

Time domain specifications

Solution of Example

- percent overshoot is:

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.16 = 16\%$$

- peak time is:

$$T_p = \frac{\pi}{\omega_d} = 3.627$$

$$\beta = \tan^{-1}\frac{\omega_d}{\xi\omega_n} = 1.047,$$

$$T_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 1.047}{0.866} = 2.42 \text{ sec},$$

$$T_s = \frac{4}{\xi\omega_n} = 8 \text{ sec},$$

$$e_{ss} = 1 - \lim_{s\to 0} sY(s) = 1 - \lim_{s\to 0} s\frac{1}{s(s^2+s+1)} = 0$$

Time domain specifications

- As noticed, the step response performance indices are functions of (ξ, ω_n).
  Also (ξ, ω_n) determine system poles:

$$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

- Therefore, the **response** of a system is determined by the position of its **poles**.

placing the closed-loop poles at **good** locations, we can shape the response of the system and achieve desired time response characteristics.

- Although the previous analysis is conducted for 2^{nd} order continuous-time system, the approach is also applicable for higher-order systems.

Thanks for your attention.

Questions?

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