CSE302 Automatic Control Engineering

Lecture 3: Mathematical Modeling of Physical Systems

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Lecture: 3

Mathematical Modeling of Physical Systems

- Modeling of Electrical Systems
- Modeling of Mechanical Systems
- Modeling of Electromechanical Systems

Mathematical Modeling of Physical Systems

Modeling

the process of representing the behavior of a real system by a collection of mathematical equations and logic.

- Models are cause-and-effect structures
  - they accept external information and process it with their logic and equations to produce one or more outputs.
  - Parameter is a fixed-value unit of information
  - Signal is a changing-unit of information
- Models can be text-based programming or pictorial representations

Types of Systems

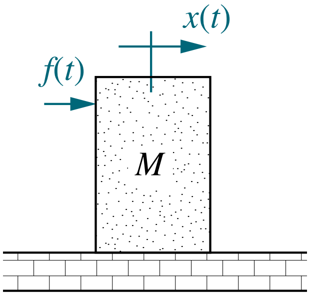
- **Static System**: If a system does not change with time.
- **Dynamic System**: if its current output depends on the past history and the present values of the input variables.

$$y(t) = \Phi[u(\tau)], \quad 0 \leq \tau \leq t$$

- Physical laws are used to predict the behavior (both static and dynamic) of systems.
  - Electrical engineering relies on **Ohm** and **Kirchoff** laws
  - Mechanical engineering on **Newton** laws
  - Electromagnetics on **Faraday** and **Lenz** laws
  - Fluids on continuity and **Bernoulli** laws
- **Example**: The displacement $x(t)$ of a a moving mass M due to an external force $f(t)$ can be modeled as:

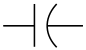
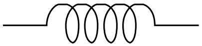
$$\mathit{Force} = \mathit{Mass} \times \mathit{Acceleration}$$

$$f(t) = M\,a = M\,\frac{d^2x(t)}{dt^2}$$



Modeling of Electrical Systems

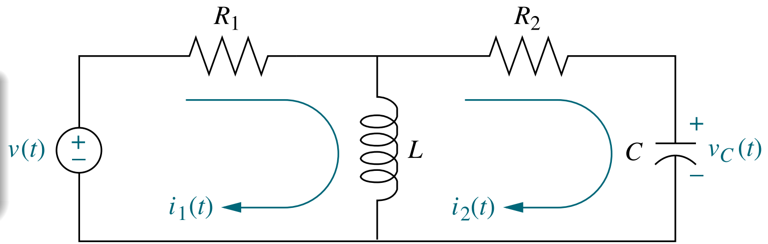
Voltage-current, voltage-charge, and impedance relationships for basic electrical components

Component	**Voltage-current**	**Current-voltage**	**Voltage-charge**	**Impedance** $Z(s) = V(s)/I(s)$	**Admittance** $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

$v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Example

Given the network, find the transfer function, $I_2(s)/V(s)$.



- Laplace transform of circuit variables, assuming zero initial conditions is shown next.

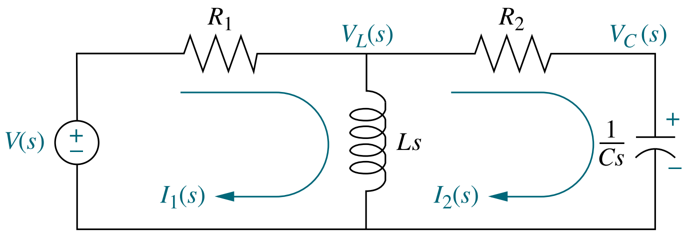
- summing voltages around each mesh:

$$R_1 I_1(s) + Ls\,[I_1(s) - I_2(s)] = V(s)$$

$$Ls\,[I_2(s) - I_1(s)] + R_2 I_2(s) + \frac{1}{Cs} I_2(s) = 0$$

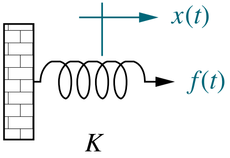
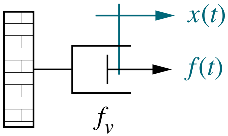
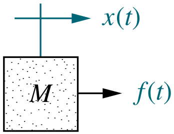
- solving for $I_2(s)$ gives:

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{LC\,s^2}{(R_1 + R_2)LC\,s^2 + (R_1 R_2 C + L)s + R_1}$$



Modeling of Mechanical Systems

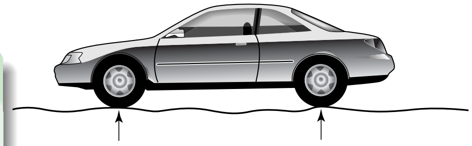
Force-velocity, force-displacement, and impedance **translational** relationships for basic mechanical components

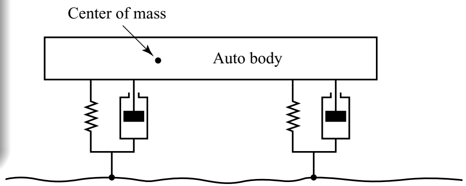
	Spring	Viscous damper	Mass
Component			
Force-velocity	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = f_v v(t)$	$f(t) = M\dfrac{dv(t)}{dt}$
Force-displacement	$f(t) = Kx(t)$	$f(t) = f_v\dfrac{dx(t)}{dt}$	$f(t) = M\dfrac{d^2x(t)}{dt^2}$
Impedence $\boldsymbol{Z_M(s) = F(s)/X(s)}$	K	$f_v s$	Ms^2

Example

a schematic diagram of an automobile suspension system is shown next. A very simplified version of the suspension system is also given.

Assuming that the motion x_i at point P is the input to the system and the vertical motion x_o of the body is the output, obtain the transfer function $X_o(s)/X_i(s)$





*

*The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.

Modeling of Mechanical Systems

Example

- The equation of motion for the system is:

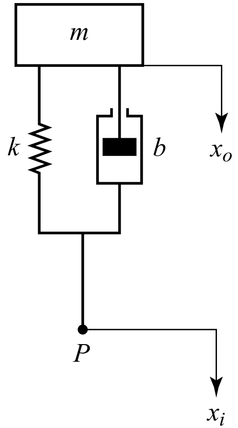
$$m\ddot{x}_o(t) + b\left[\dot{x}_o(t) - \dot{x}_i(t)\right] + k\left[x_o(t) - x_i(t)\right] = 0$$

- Taking the Laplace transform of the equation of motion[a], we obtain:

$$m\,s^2\,X_o(s) + b\,s\left[X_o(s) - X_i(s)\right] + k\left[X_o(s) - X_i(s)\right] = 0$$
$$m\,s^2\,X_o(s) + b\,sX_o(s) + kX_o(s) = b\,sX_i(s) + kX_o(s) - X_i(s)$$

- Hence the transfer function Xo(s)/Xi(s) is given by:

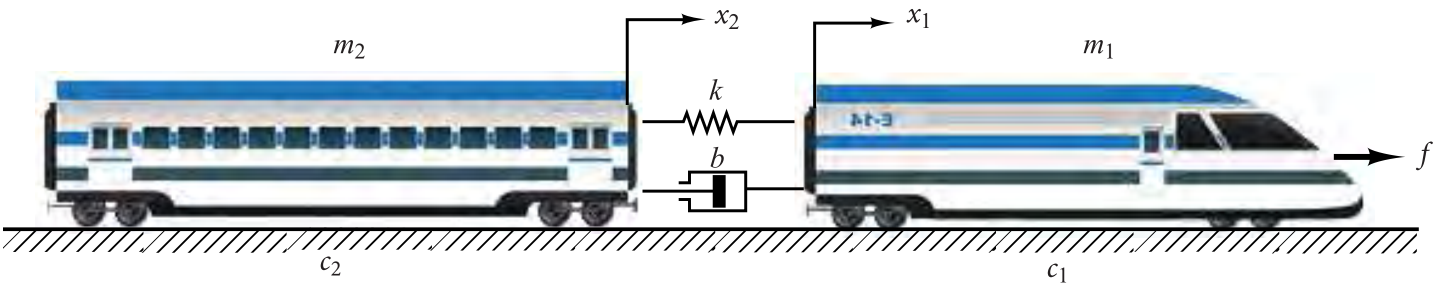
$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms2 + bs + k}$$



[a] assuming zero initial conditions

Example

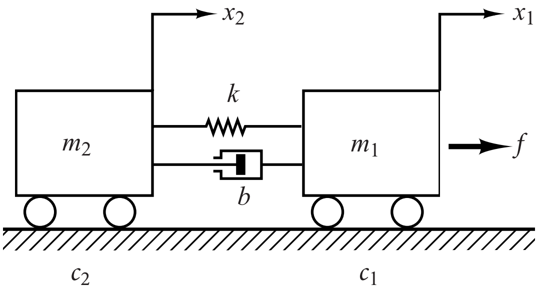
Consider the two carriage train system shown. obtain the transfer function $X_1(s)/F(s)$



- The two carriage train is schematically represented as:

  - Forces acting on carriages 1 and 2, respectively:

$$m_1\ddot{x}_1 + b(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) + c_1\dot{x}_1 = f$$
$$m_2\ddot{x}_2 + b(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) + c_2\dot{x}_2 = 0$$



- Taking Laplace transform and rearranging:

$$\left[m_1\, s^2 + (b + c_1)s + k\right] X_1(s) - [b\, s + k]\, X_2(s) = F(s)$$
$$-[b\, s + k]\, X_1(s) + \left[m_2\, s^2 + (b + c_2)s + k\right] X_2(s) = 0$$

Modeling of Mechanical Systems

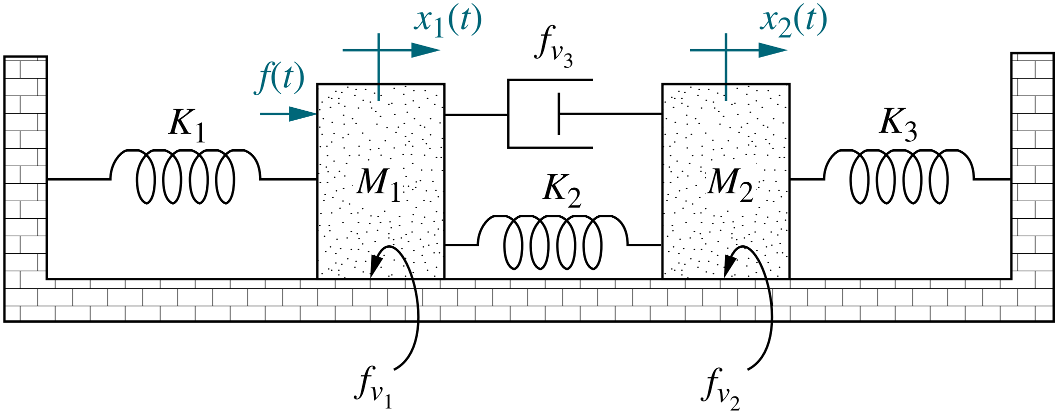
Example

- Solving the previous two equations, gives transfer function (with F as input and X_1 as output):

$$\frac{X_1(s)}{F(s)} = \frac{m_2\, s^2 + (b + c_2)s + k}{m_1 m_2 s^4 + m_1(2b + c_1 + c_2)s^3 + \left[k(m_1 + m_2) + b(c_1 + c_2) + c_1 c_2\right] s^2 + k(c_1 + c_2)s}$$

- Note: Transfer function is a frequency domain equation that gives the relationship between a **specific input** to a specific output

Find the transfer function, $X_2(s)/F(s)$, for the system:

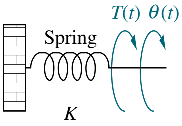
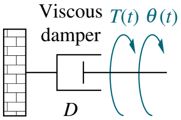
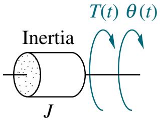


Note that friction shown here is viscous friction. Thus, fv_1 and fv_2 are not Coulomb friction, but arise because of a **viscous** interface.

Modeling of Mechanical Systems

Rotational Mechanical System

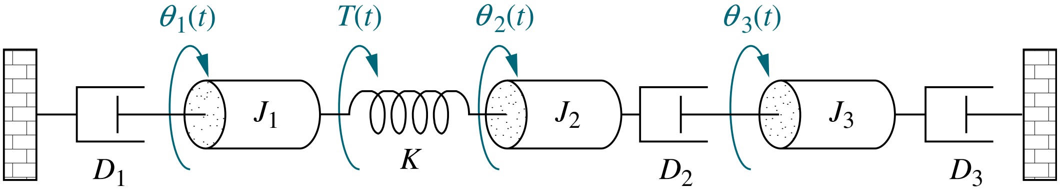
Torque-angular velocity, torque-angular displacement, and impedance **rotational** relationships for basic mechanical components

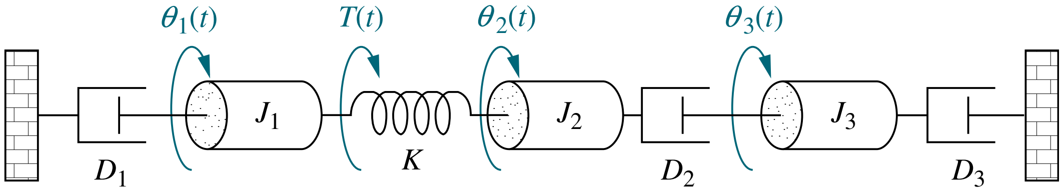
Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

$T(t)$ – N-m (newton-meters), $\theta(t)$ – rad (radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton-meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

Example

Write the Laplace transform of the **equations of motion** for the system shown





$$\left(J_1 s^2 + D_1 s + K\right)\ \theta_1(s) - K\theta_2(s) - 0\ \theta_3(s) = T(s)$$

$$-K\theta_1(s) + (J2s^2 + D_2 s + K)\,\theta_2(s) - D_2\, s\,\theta_3(s) = 0$$

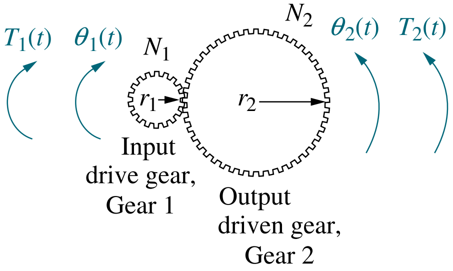
$$-0\ \theta_1(s) - D_2 s\theta_2(s) + \left(J_3 s^2 + D_3 s + D_2 s\right)\theta_3(s) = 0$$

- gears allow to match the drive system and the load
- a trade–off between speed and torque
- mechanical impedances is reflected through gear trains by:

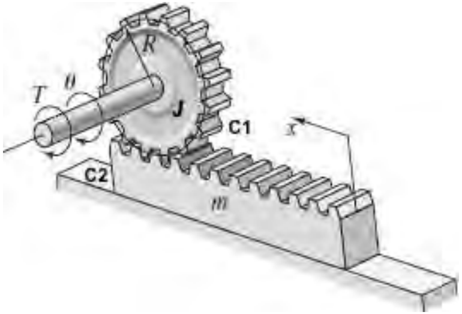
$$\left(\frac{\text{Number of teeth of gear on } \textbf{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textbf{source} \text{ shaft}}\right)^2$$

- interaction between two gears:

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}, \qquad \frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}, \qquad \frac{Z_2}{Z_1} = \left(\frac{N_2}{N_1}\right)^2$$



- with rack and pinion system, the **rotational** motion is transformed **into transitional** motion.
- Relation between input rotational torque and output linear velocity[§]:



$$T_{in} - T_{out} = J\frac{d\omega}{dt} + c_1\,\omega \qquad \text{rotational equation}$$

$$F - c_2\,v = m\frac{dv}{dt} \qquad \text{transitional equation}$$

$$T_{out} = r\,F, \quad \omega = v/r$$

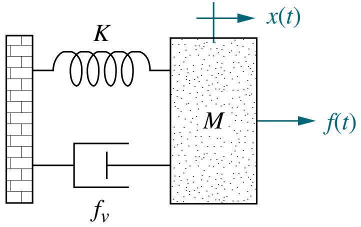
- manipulating the rotational and transitional equations, we get:

$$T_{in} = \left(\frac{c_1}{r} + c_2\ r\right) v + \left(\frac{J}{r} + m\ r\right)\frac{dv}{dt}$$

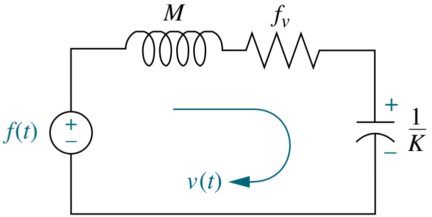
[§] For simplicity, the spring effects are ignored

Analogy Approach

quantity		equivalent	
mass	M	inductor	M henries
viscous damper	f_v	resistor	f_v ohms
spring	k	capacitor	$1/k$ farads
applied force	$f(t)$	voltage source	$f(t)$
velocity	$v(t)$	mesh current	$v(t)$



	Flow Variable (FV)	Potential Variable (PV)
Electrical	Current	Voltage
Mech. Transitional	Force	Velocity
Mech. Rotational	Torque	Angular Velocity
Hydraulic	Volumetric Flow Rate	Pressure
Pneumatic	Mass Flow Rate	Pressure
Thermal	Heat Flow Rate	Temperature



Analogy Approach

- Based on electrical analogies, we can derive the fundamental equations of systems in five disciplines of engineering:
  - Electrical, Mechanical, Electromagnetic, Fluid, and Thermal.
- By using this analogy method to first derive the fundamental relationships in a system, the equations then can be represented in block diagram form, allowing secondary and nonlinear effects to be added.
  - This two-step approach is especially useful when modeling large coupled systems.

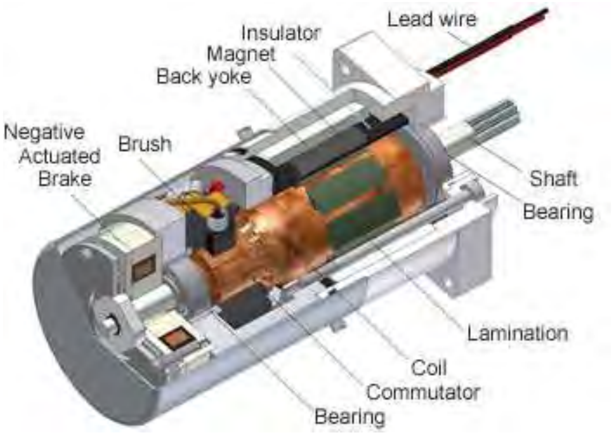
Electromechanical Systems

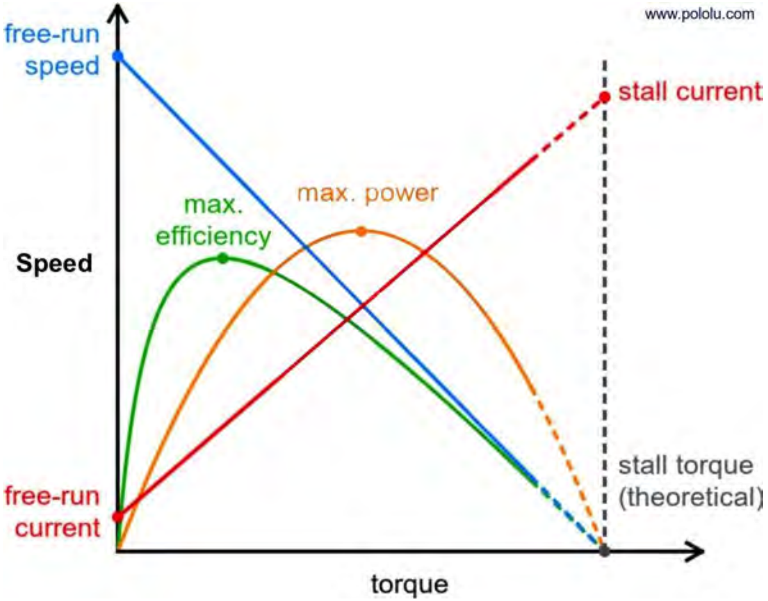
- Combine electrical and mechanical processes.
- Devices carrying out electrical operations using moving parts are known as electromechanical.
  - Relays, Solenoids, Electric Motors, Electric Generators, Switches, etc.

Electromechanical Systems

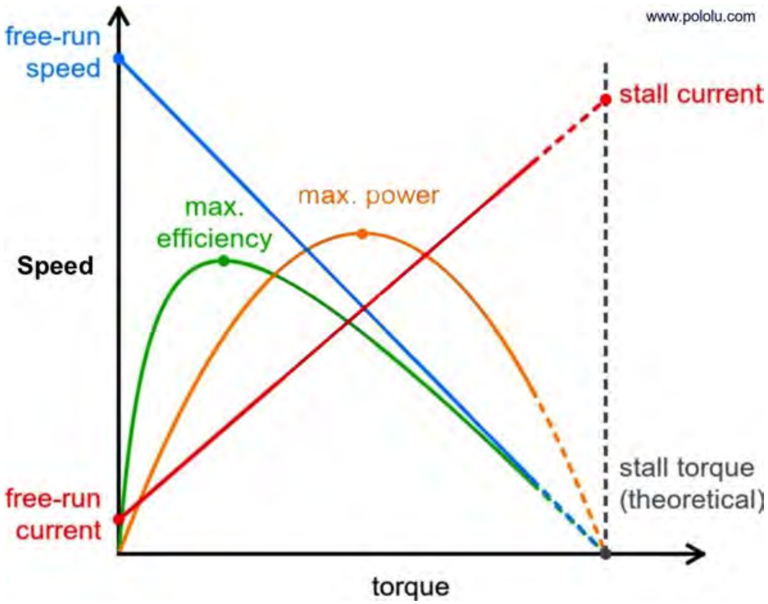
DC Motors

- Speed control through variable voltage applied to the armature terminals
- each motor has a specific Torque/Speed curve and Power curve.

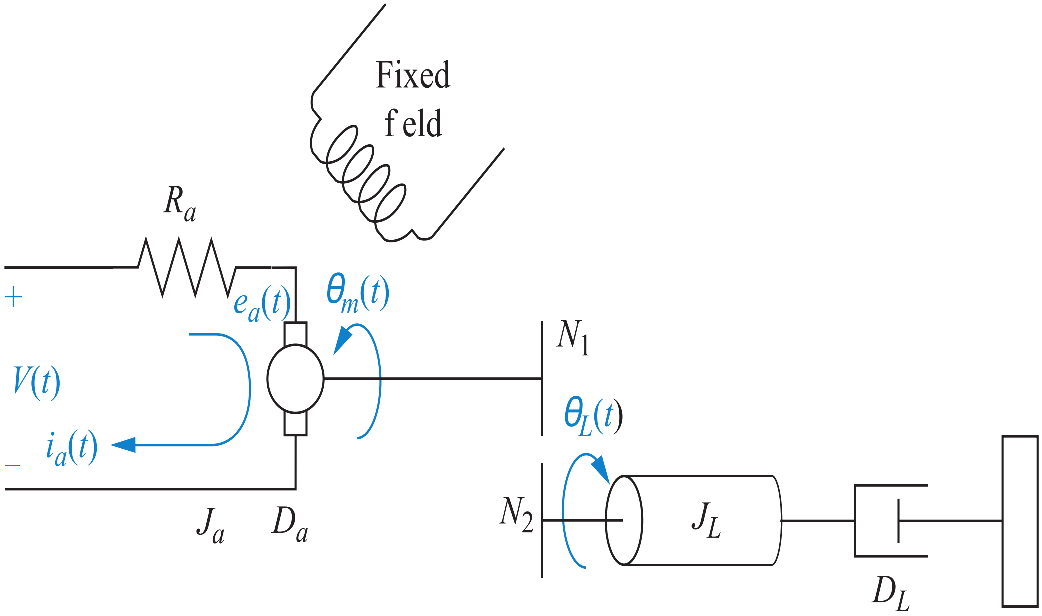




- Torque is **inversely** proportional to output shaft speed.
- Motor characteristics are frequently given as two points on this graph:
  - **stall torque**: the point on at which the torque is maximum, but the shaft is not rotating.
  - **no load speed**: maximum output speed of the motor.



DC Motor with Load



DC Motor Model

- for a DC motor, mechanical and electrical equations are:

$$V = R\,i + L\frac{di}{dt} + e_a \qquad (1)$$

$$e_b = K_t\,\omega \qquad (2)$$

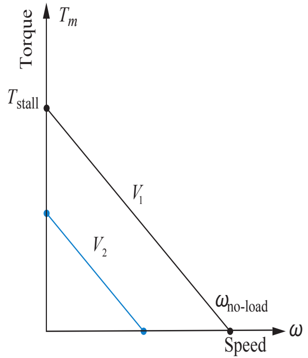
$$T = K_t\,i = J_m\frac{d\omega}{dt} + D_m\,\omega \qquad (3)$$

T	motor torque
K_t	torque constant
i	current,
V	supplied voltage,
ω	rotor speed,
e_b	back-emf ($e_b = K_e\,\omega$),
R, L	resistance and induction

- For a fixed voltage, torque–speed curves are derived from (1) & (3):

$$T = \frac{k_t}{R}(V - K_t\,\omega) = \frac{k_t}{R}\,V - k_m^2\omega$$

- $K_m = k_t/\sqrt{R}$ is the **motor constant**, [numerically, $kt = ke$]
- slope of the torque–speed curves is $-K_m^2$
- voltage-controlled DC motor has inherent damping in its mechanical behavior
- torque increases in proportion to applied voltage and reduces as ω increases.

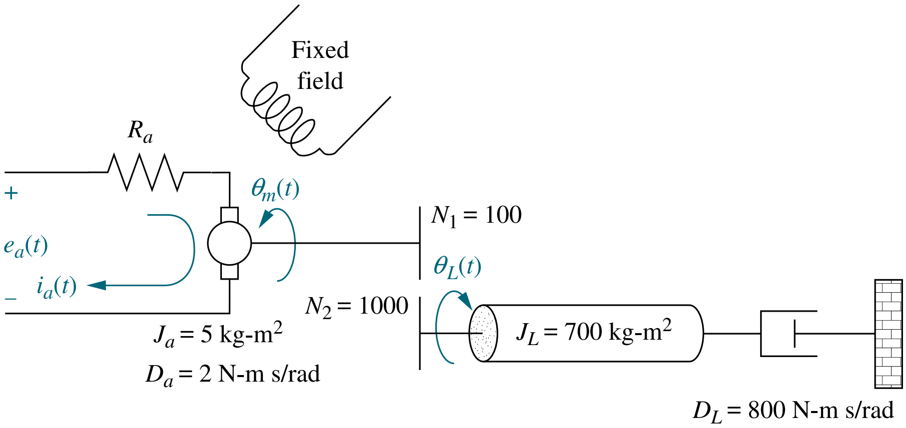


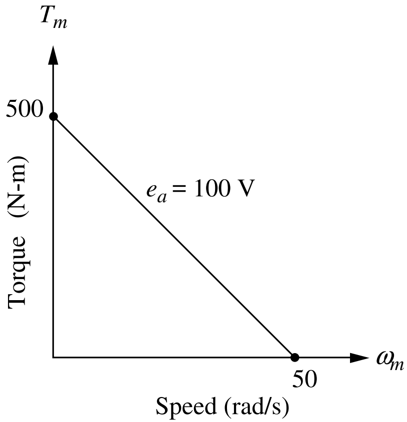
DC Motor Model

Example

Example

Given the DC motor with load system and torque-speed curve, find the transfer function, $\theta_L(s)/V(s)$.





DC Motor Model

Example

- to get the transfer function, we combine Laplace transforms of (1) through (3) and simplifying:

$$\frac{\theta_m(s)}{V(s)} = \frac{k_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}$$

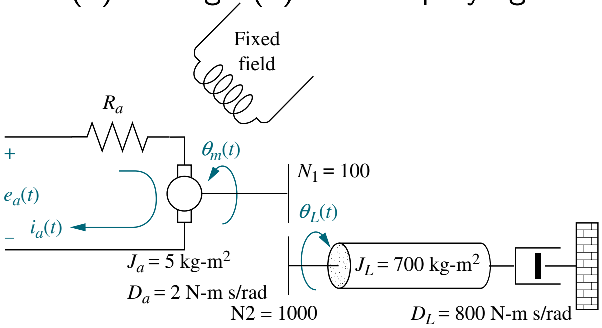
- **total** inertia and damping **at motor armature**:

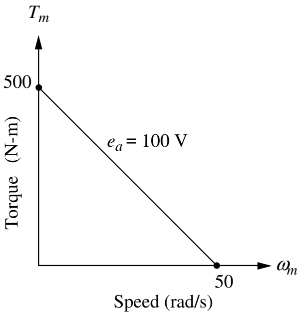
$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12$$

$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10$$

- From the torque–speed curve:

$$T_{stall} = 500, \qquad \omega_{no-load} = 50, \qquad V = 100$$





DC Motor Model

Example

- Hence the electrical constants, K_t/R_a and K_b are:

$$\frac{Kt}{Ra} = \frac{Tstall}{V} = \frac{500}{100} = 5, \qquad Kb = \frac{V}{\omega_{no-load}} = \frac{100}{50} = 2$$

- Substituting system parameters into Eq.(5) yields:

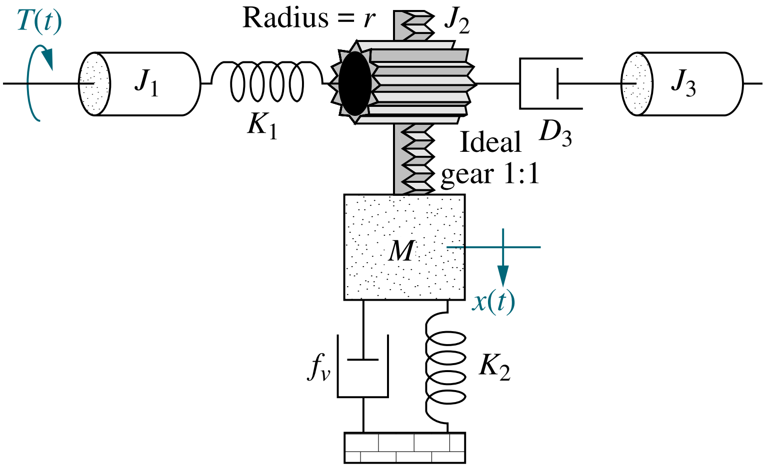
$$\frac{\theta_m(s)}{V(s)} = \frac{5/12}{s\left[s + \frac{1}{12}(10+52)\right]} = \frac{0.417}{s(s+1.667)}$$

- to find the final transfer function (from the load–side, i.e. $\theta_L/V(s)$), we use the gear ratio, $N_1/N_2 = 1/10$, hence we get:

$$\frac{\theta_L(s)}{V(s)} = \frac{0.0417}{s(s+1.667)}$$

Example

Given the combined translational and rotational system shown, find the transfer function, $G(s) = X(s)/T(s)$.



Electromechanical Systems

Example

- Writing the equations of motion,

$$\begin{aligned}(J_1\ s^2 + K_1)\ \theta_1(s) - K_1\ \theta_2(s) &= T(s)\\ -K_1\ \theta_1(s) + (J_2\ s^2 + D_3\ s + K_1)\ \theta_2(s) + F(s)\ r - D_3\ s\ \theta_3(s) &= 0\\ -D_3\ s\ \theta_2(s) + (J_2\ s^2 + D_3\ s)\ \theta_3(s) &= 0\end{aligned}$$

  where $F(s)$ is the opposing force on J_2 due to the translational member and r is the radius of J_2.

- for the translational member,

$$F(s) = (M\ s^2 + f_v\ s + K_2)X(s) = (M\ s^2 + f_v\ s + K2)\ r\ \theta(s)$$

- Substituting $F(s)$ back into the second equation of motion:

$$
\begin{aligned}
(J_1\ s^2 + K_1)\ \theta_1(s) - K_1\ \theta_2(s) &= T(s) \\
-K_1\ \theta_1(s) + \left[(J_2 + M\ r^2)\ s^2 + (D_3 + f_v\ r^2)\ s + (K_1 + K_2\ r^2)\right]\ \theta_2(s) - D_3\ s\ \theta_3(s) &= 0 \\
-D_3\ s\ \theta_2(s) + (J_2\ s^2 + D_3\ s)\ \theta_3(s) &= 0
\end{aligned}
$$

  Note: the translational components were **reflected** as equivalent rotational components by the **square of the radius**.

- Solving for $\theta_2(s)$,

$$\theta_2(s) = \frac{K_1(J_3 s^2 + D_3 s)T(s)}{\Delta}$$

  Δ is the determinant formed from the coefficients of the three equations of motion.

- Since $X(s) = r\theta_2(s)$, then:

$$\frac{X(s)}{T(s)} = \frac{r\ K_1(J_3\ s^2 + D_3\ s)}{\Delta}$$

Thanks for your attention.

Questions?

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