CSE302 Automatic Control Engineering

Lecture 2: Mathematical Modeling of Control Systems

Asst. Prof. Dr.Ing.

Mohammed Nour A. Ahmed

mnahmed@eng.zu.edu.eg

https://mnourgwad.github.io

Zagazig University | Faculty of Engineering | Computer and Systems Engineering Department | Zagazig, Egypt

Lecture: 2

Mathematical Modeling of Control Systems

- Mathematical Models
- Differential Equation Model
- Block Diagram Representation

Mathematical Models

Mathematical Model

A set of mathematical equations (e.g., differential equations) that describes the input-output behavior of a system.

- These models are useful for **simulation**, prediction/forecasting, design/performance evaluation, and **analysis and design** of control systems.

Lumped vs. Distributed Systems

- lumped system has a finite number of state variables
- distributed system has an infinite number of state variables

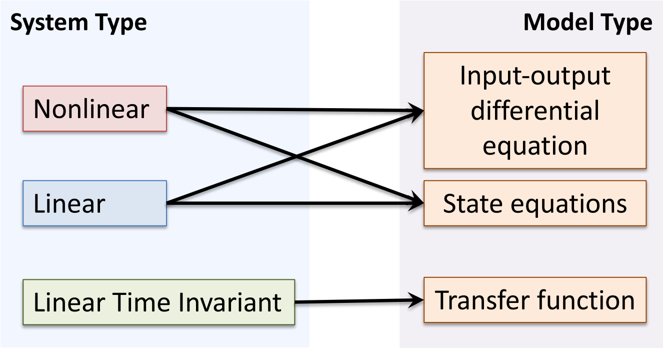
state variables

a set of variables whose values at any moment completely characterize the system

Mathematical Models

There are different types of lumped-parameter models. The mostly used:

- Differential equation model (for linear and nonlinear systems)
- Transfer function model (for linear time invariant systems)
- State space model (for linear and nonlinear, SISO, and MIMO systems)



Differential Equation Model

a time domain mathematical model of control systems.

- For linear systems, we can often represent the system dynamics through an n^{th} order ordinary differential equation (ODE):

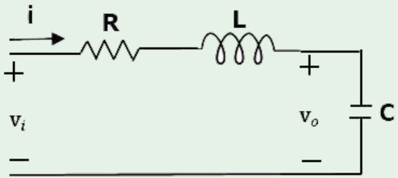
$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + a_2 y^{(n-2)}(t) + \cdots + a_{n-1}\dot{y}(t) + a_n y(t) =$$
$$b_0 u^{(m)}(t) + b_1 u^{(n-1)}(t) + b_2 u(b-2)(t) + \cdots + b_{n-1}\dot{u}(t) + b_m u(t)$$

- Input: $u(t)$; Output: $y(t)$, system parameters: $a_1, \cdots, a_n; \quad b_0, \cdots, b_m$
- The $y^{(k)}$ notation means we're taking the k^{th} derivative of $y(t)$
- system order is the order of the ODE
- Typically, $m > n$

Differential Equation Model

Example

Derive the model of the following series RLC circuit with input voltage applied to circuit v_i and voltage across the capacitor, v_o as output.



- Mesh equation: $v_i = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + v_o$
- Substituting with capacitor current $i = c\frac{\mathrm{d}v_o}{\mathrm{d}t}$

$$v_i = RC\frac{\mathrm{d}v_o}{\mathrm{d}t} + LC\frac{\mathrm{d}^2v_o}{\mathrm{d}t^2} + v_o \quad \Rightarrow \quad \frac{\mathrm{d}^2v_o}{\mathrm{d}t^2} + \left(\frac{R}{L}\right)\frac{\mathrm{d}v_o}{\mathrm{d}t} + \left(\frac{1}{LC}\right)v_o = \left(\frac{1}{LC}\right)v_i$$

- The above equation is a second order **differential equation**.

Transfer Function

Transfer Function

the ratio of Laplace transform of output and Laplace transform of input by assuming all the initial conditions are zero.

$$\mathscr{L}\{u(t)\} = U(s), \quad \mathscr{L}\{y(t)\} = Y(s), \quad \mathscr{L}: \text{ is the Laplace operator}$$

$$\text{Transfer Function} = \left.\frac{\mathscr{L}\{y(t)\}}{\mathscr{L}\{x(t)\}}\right|_{IC=0} = \frac{Y(s)}{X(s)}$$

- Given that ODE description, we can take the Laplace Transform:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

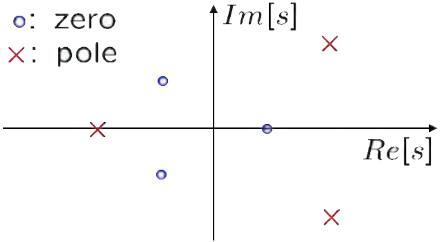
- denominator polynomial order $>$ numerator polynomial order, transfer function is said to be **proper**. Otherwise **improper**

Transfer Function

- the transfer function $H(s)$ of the system is given as:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

- s is a complex variable (**complex frequency**) and is given as: $s = \delta + j\omega$
- Roots of **numerator** are called the **zeros**
- Roots of the **denominator** are called the **poles**
- pole–zero plot:



Transfer Function

Example

- Given: $H(s) = \dfrac{2s + 1}{s^3 - 4s^2 + 6s - 4}$
- Zeros: z1 = -0.5
- Poles: solve s3 -4s2 +6s -4 = 0,
- use MATLAB `roots` command:

```
> poles = roots[1 -4 6 -4]
poles =
       2,
       1 + j, 1 - j
```

- Factored form:

$$H(s) = \frac{2s + 0.5}{(s - 2)(s - 1 - j)(s - 1 + j)}$$

Analyzing Generic Physical Systems

Seven-step algorithm:

1. Identify dynamic variables, inputs (u), and system outputs (y)
2. Focus on one component, analyze the dynamics (physics) of this component
   - How? Use Newton's Equations, KVL, or thermodynamics laws, etc.
3. After that, obtain an nth order ODE:

$$\sum_{i=1}^{n} \alpha_i \, y^{(i)}(t) = \sum_{j=1}^{m} \beta_j \, u^{(j)}(t)$$

4. Take the Laplace transform of that ODE
5. Combine the equations to eliminate internal variables
6. Write the transfer function from input to output
7. For a certain control $U(s)$, find $Y(s)$, then $y(t) = \mathscr{L}^{-1}\{Y(s)\}$

- For the following transfer functions, determine
  - Whether the transfer function is proper or improper
  - Poles of the system
  - zeros of the system
  - Order of the system

a) $G(s) = \dfrac{s+3}{s(s+2)}$

b) $G(s) = \dfrac{s}{s(s+1)(s+2)(s+3)}$

c) $G(s) = \dfrac{(s+3)^2}{s(s^2+10)}$

d) $G(s) = \dfrac{s^2(s+1)}{s(s+10)}$

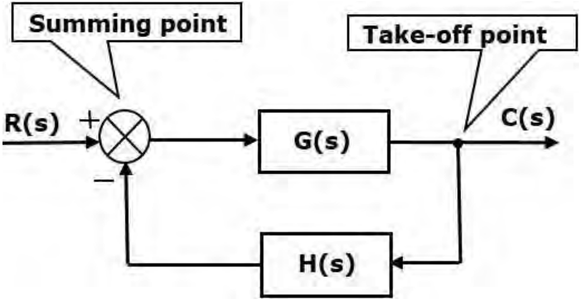
	proper/impr.	Poles	zeros	Order
a)				
b)				
c)				
d)				

Block Diagram

Representation of Control Systems

Block Diagram

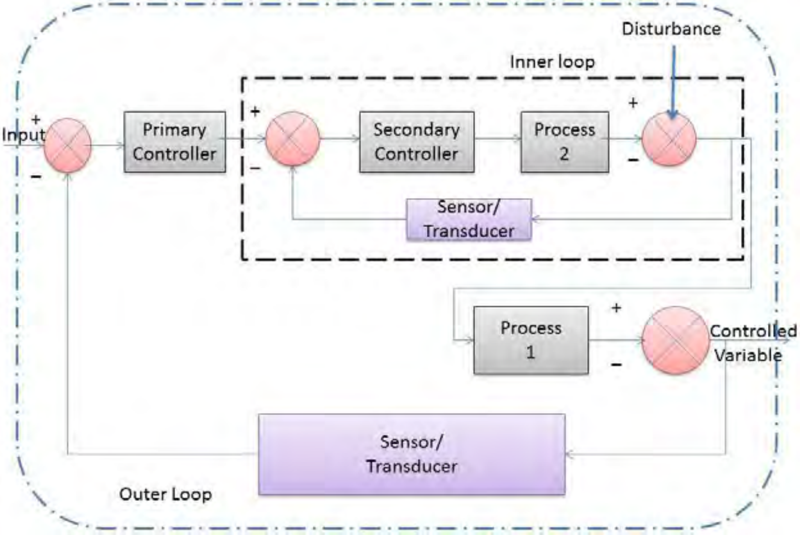
a shorthand pictorial representation of the cause-and-effect relationship of a system.



- the block usually contains a description of or the name of the element, gain, or the symbol for the mathematical operation to be performed on the input to yield the output.
- The arrows represent the direction of information or **signal flow**.

Examples of Block Diagrams

Temperature Control System



Importance of Block Diagrams

- Graphical representation of interconnected systems are important
- A system may consist of multiple subsystems: the output of one may be the input to another, and so on
- Each subsystem is represented by a functional block, labeled with the corresponding transfer function
- Blocks are connected by arrows to indicate signal flow directions

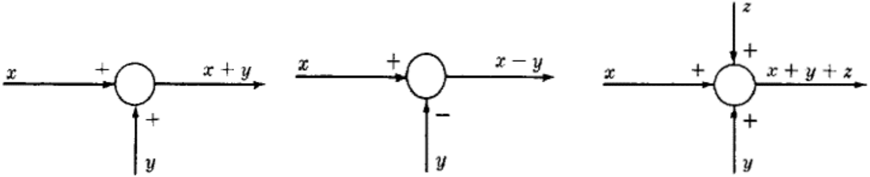
Advantages:

- Easy for visualization purpose
- Can represent a class of similar systems
- Most importantly: can infer overall relationship between inputs and outputs, and hence analyze the system stability and performance

Block Diagram Building Blocks

Summing Point

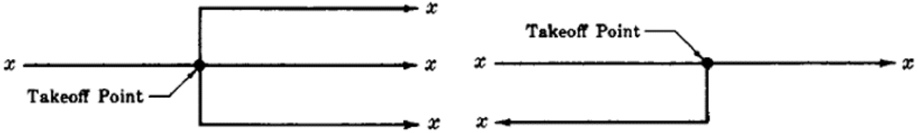
- indicates the operations of addition and subtraction
- represents by a circle (with or without X inside) with the appropriate plus or minus sign associated with arrows into the circle.
- The output is the **algebraic sum** of the inputs.
- Any number of inputs may enter a summing point.



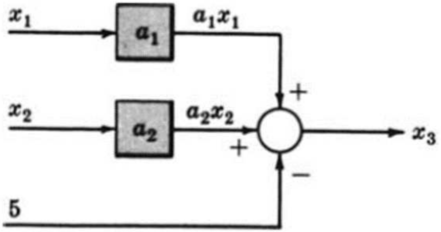
Block Diagram Building Blocks

Takeoff Point

- used for signal branching to have the same signal or variable as input to other blocks
- This permits the signal to proceed unaltered along several different paths to several destinations.



- Consider the following block diagram in which x_1, x_2, x_3 , are variables, and a_1, a_2 are transfer functions:



The output x_3 is calculated as:

$$x_3 = a_1\, x_1 + a_2\, x_2 + 5$$

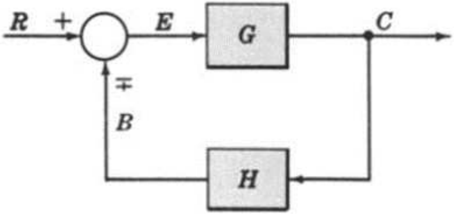
Block Diagram Building Blocks

Example/Quiz

- Draw the Block Diagrams of the following equations.

$$x_2 = a_1 \frac{dx_1}{dt} + \frac{1}{b} \int x_1 \, dt$$

$$x_3 = a_1 \frac{d^2 x_1}{dt^2} + 3 \frac{dx_1}{dt} - b x_1$$



- Tracking error: $E(s) = R(s) - B(s)$
- Forward/Direct transfer function (FTF): $\frac{C(s)}{E(s)} = G(s)$
- Loop transfer function (LTF): $\frac{B(s)}{E(s)} = G(s)H(s)$
- Closed-loop transfer function (CLTF): $\frac{C(s)}{R(s)} =??$

Important Definitions

Characteristic Equation

- The **control ratio** is the closed loop transfer function of the system.
- The denominator of closed loop transfer function determines the system **characteristic equation** as:

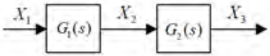
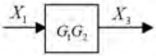
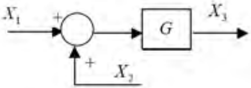
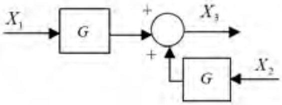
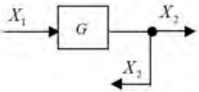
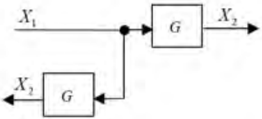
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

$$1 \pm G(s)H(s) = 0$$

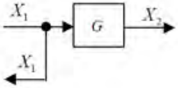
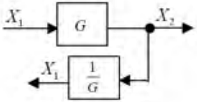
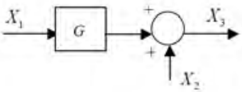
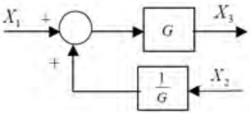
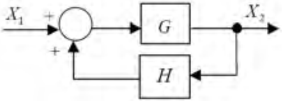
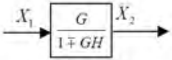
Block Diagram Algebra

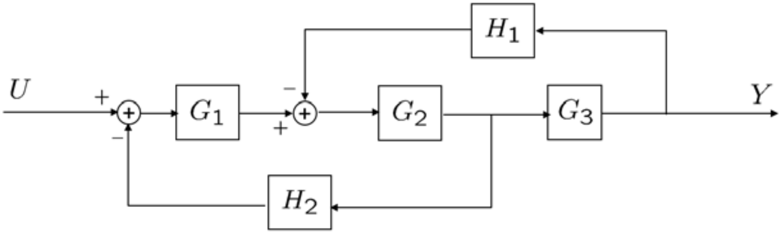
Reduction techniques

Block Diagram Transformations [Taken from Dorf & Bishop Textbook]

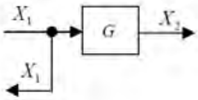
Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		

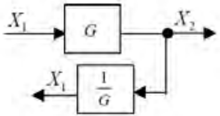
Block Diagram Transformations [Taken from Dorf & Bishop Textbook]

Transformation	Original Diagram	Equivalent Diagram
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		



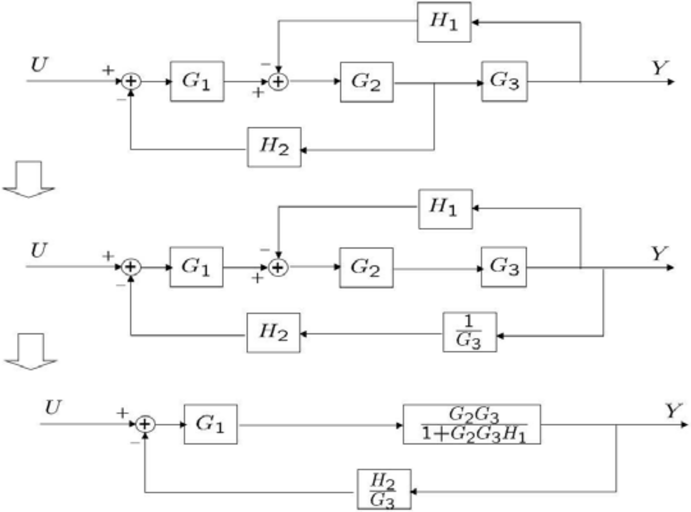
- Find the CLTF utilizing the previous transformations
- Hint: use property 4 (see previous slide)
- Property 4: sliding a branch point past a function block

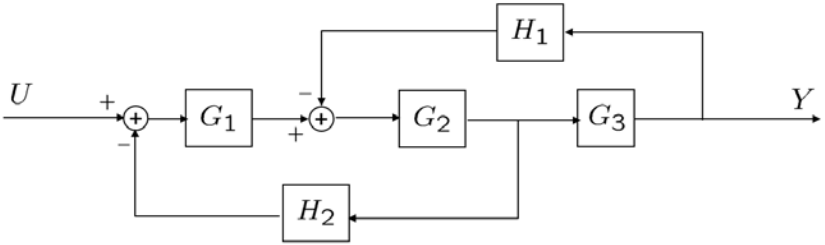




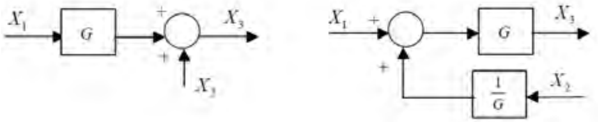
Block Diagram Algebra

Examples



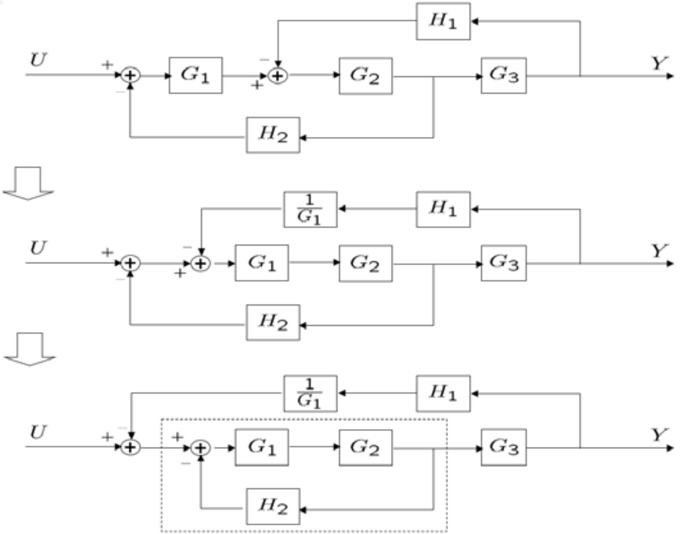


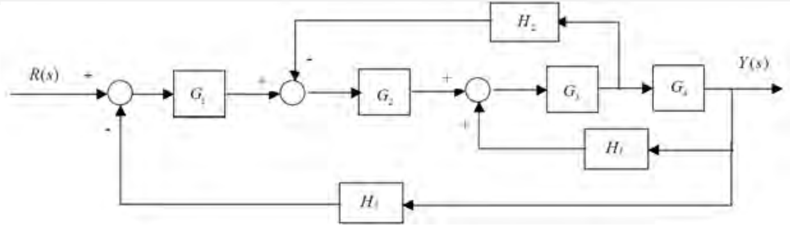
- Can we use another property?
- Yes, we can use Property 5 (moving a summing point ahead of a block)



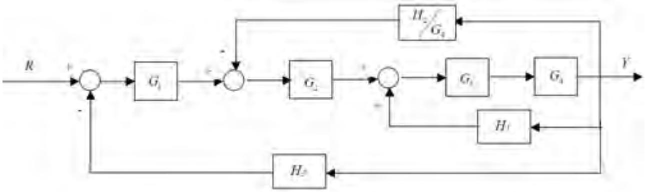
Block Diagram Algebra

Examples



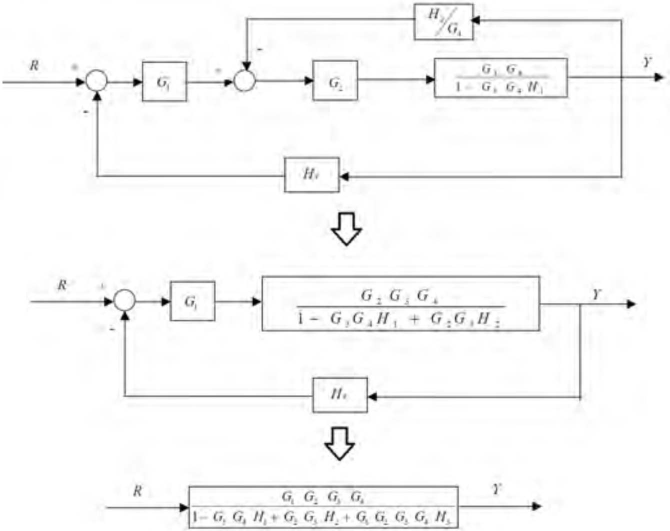


- **Solution:** First, let's move H_2 behind block G_4 so that we can isolate the $G_3 - G_4 - H_1$ feedback loop
- Again, we use Property 4 to get:



Block Diagram Algebra

Examples



Thanks for your attention.

Questions?

Asst. Prof. Dr.Ing.
Mohammed Nour A. Ahmed
mnahmed@eng.zu.edu.eg
https://mnourgwad.github.io

Robotics Research Interest Group (zuR^2IG)
Zagazig University | Faculty of Engineering | Computer and Systems Engineering Department | Zagazig, Egypt